Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Notations and Assumptions

I Logic and conventions

i The iff convention

iff is a shorthand for "if and only if". Splitting **iff** into *if-then* clauses shows that it is a natural-language version of the logical equivalence $\iff$. All definitions are understood to be **iff** clauses, which is consistent with the fact that every definition expresses an equivalence.

ii The definition symbol

Given variables a and b, $\mathrm{a} \triangleq \mathrm{b}$ is the predicate $\mathrm{a} = \mathrm{b}$ with the extra information that a and b are *assumed* to be equal.

iii Iverson brackets

Given a Boolean expression ϕ, $\left[\phi\right]$ encodes the truth value of ϕ as a `bit`:

$$\left[\phi\right] \triangleq \begin{cases} 0 & (\phi \text{ is false}) \\ 1 & (\phi \text{ is true}). \end{cases} \tag{1}$$

For example, $\left[1 > 0\right] = 1$ but $\left[\sqrt{2} \in \mathbf{Q}\right] = 0$. For interpretations of Boolean operators in logic, see [1].

iv Conventions

Variables

To distinguish a variable from a value within its range, we may write variables in italics. For instance, x denotes a variable, whereas the predicates $\mathrm{x} = 1$ and $\mathrm{x} \in \mathrm{X}$ are about *values* x.

Copies of variables

Given variables, say v and x, we implicitly define $v', v_x, w, \ldots, z$ as *copies* of v. Thus,

(2) $v', v'', \ldots$ (scope: no conflict with another meaning of prime, *e.g.*, the derivative),

(3) $v_x, v_{x'}, \ldots$ (scope: no conflict with a prior index),

(4) $v, w, \ldots, z, z', \ldots$ (scope: no conflict with a prior definition, *e.g.*, $\mathrm{y} = 1/\mathrm{x}$)

range over V when $\mathrm{v} \in \mathrm{V}$. The $\forall$ quantifier of a copy, *e.g.*, $\forall \mathrm{v}' \in \mathrm{V}$, applies where the copy first appears. Unless assigned otherwise, i, j, k, l, m, n implicitly denote natural numbers, and α, β, γ implicitly stand for scalars. Similarly, t is implicitly a real "time parameter". Case matters. To avoid ambiguity, the complex numbers i and j may be set in bold. If a variable f is a function of x, then copying f copies every variable of f. For instance, given the parametric expression $\mathrm{y} = \mathrm{f}(\mathrm{x})$, $\mathrm{f_i}$ is implicitly a function of x_i, and y_i is a copy of y so that $\mathrm{y_i} = \mathrm{f_i}(\mathrm{x_i})$.

Explicit dependence on variables

Sometimes, for the sake of expressivity, we wish to make explicit that a quantity η depends on $x, \ldots, \varepsilon, \ldots$ by writing

$$\eta = \eta(\mathrm{x}, \ldots, \varepsilon, \ldots), \tag{5}$$

which reads as "η as a function of $x, \dots, \varepsilon, \dots$". For instance, in the following definition of continuity for f:

(6) For all x, for all $\varepsilon > 0$, there exists $\eta > 0$ such that $|x - x'| < \eta$ implies $|f(x) - f(x')| < \varepsilon$,

the clause "there exists $\eta > 0$" can be written as "there exists $\eta = \eta(x, \varepsilon) > 0$", to express that the η we address is parametrized by x and ε. In contrast, *uniform* continuity removes the dependence on x, which reads as $\eta = \eta(\varepsilon)$.

Index notation

As with sequences, where v_n is preferred over the functional notation $v(n)$, we use index notation when readability is at stake. For instance, $\eta = \eta(\varepsilon)$ can be written as $\eta = \eta_\varepsilon$. Additionally, in index notation, the parentheses around multi-variable indices are omitted. To elaborate with the same example, $\eta_{(x,\varepsilon)}$ becomes $\eta_{x,\varepsilon}$.

II Sets

i Subsets and supersets

The symbols $\subseteq$ and $\supseteq$ are the standard symbols for set ordering. No specific symbol is reserved for strict ordering: whenever strict inclusion is required for $A \subseteq B$, the constraint $A \neq B$ is explicitly added.

ii Union and intersection

Given A and B, $A \cup B$ is the union of A and B. More generally, for any collection Γ, the union U of Γ is expressed as follows:

(7) $$U \triangleq \bigcup \Gamma \triangleq \bigcup_{S \in \Gamma} S.$$

The symbol $\cap$ denotes intersection in the same manner.

iii Special mappings

Given X, the identity mapping I (or id_X) is the diagonal $\{(x, x) : x \in X\}$ of $X \times X$.

iv Cardinality and equinumerosity

The cardinality of X is denoted by card(X). The fact that $X \mapsto \mathrm{card}(X)$ is not a function (card's domain is the class of sets, which is not a set) does no harm (we never deal with all sets). We did not select a global symbol to express equinumerosity between X and Y, *i.e.*, that $\mathrm{card}(X) = \mathrm{card}(Y)$. However, in the context of groups, $X \cong Y$ means that there exists an isomorphism, *i.e.*, a bijective morphism, of X onto Y. In a normed-space context, $X \equiv Y$ means that there exists a linear surjective isometry of X onto Y.

III Topological vector spaces

i Scalar field

We sometimes need to clearly specify which field we are dealing with. For instance, [12.7] of [4] has no real counterpart, and [3.2] of [4] is a purely real result. But, in most cases, the choice of the field makes no difference. So, unless the field must be explicitly given, we tacitly assume that the scalar field is **C** but that **R** could be substituted for **C**.

ii A tacit result

In a vector space, the scalar multiplication of every nonempty set S by 0 is $\{0\}$. This will always be tacitly used. The proof is given in (b) of Exercise [1.1].

iii Intervals

Given $\{a, b\} \subseteq X$, the *closed interval*, or "segment", $[a, b]$ is defined as the convex hull of $\{a, b\}$, that is, the set of all convex combinations $\lambda(t) = (1-t)a + tb$ $(0 \leq t \leq 1)$. We also define three other types of intervals, as follows:

$$]a, b[\triangleq \{\lambda(t) : 0 < t < 1\}, \tag{8}$$ (open interval)

$$]a, b] \triangleq \{\lambda(t) : 0 < t \leq 1\}, \tag{9}$$

$$[a, b[\triangleq \{\lambda(t) : 0 \leq t < 1\}. \tag{10}$$

This notation is expressive:], [looks like a window being opened with open hands 👐, and, by contrast, [,] evokes a closed window!

When $X = \mathbf{R}$, we tacitly assume that $a \leq b$, so that the preceding definitions coincide in $\mathbf{R}$ with the usual ones, which are made from $\leq$. For instance, $]a, b[= \{x : a < x < b\}$. Also in $\mathbf{R}$, unbounded intervals are defined as follows:

$$]a, \infty) \triangleq \{x : a < x\}, \tag{11}$$

$$[a, \infty) \triangleq \{x : a \leq x\}, \tag{12}$$

and

$$(-\infty, b[\triangleq \{x : x < b\}, \tag{13}$$

$$(-\infty, b] \triangleq \{x : x \leq b\}. \tag{14}$$

This time, the "(" and ")" are kept for the "closure at infinity", because $\pm\infty$ is not an element of $\mathbf{R}$.

iv Vector space bases

Given a vector space X, a subset B of X is a basis of X **iff** the following mapping

$$\begin{aligned} \Big\{(z_u)_{u\in B} : z_u \in \mathbf{C}, \{u : z_u \neq 0\} \text{ is finite}\Big\} &\to X \\ (z_u) &\mapsto \sum_{u\in B} z_u u \end{aligned} \tag{15}$$

is a bijection from all *finitely supported* families (z_u) onto X. The axiom of choice (AC) forces

(a) the existence of such B (the proof is similar to the second part of the Hahn–Banach theorem [3.2] of [4] with B playing the role of Λ);

(b) all bases to have the same cardinality, which is called the *dimension* of X and is denoted by $\dim X$.

We now turn to the finite-dimensional case over the field $\mathbf{C}$. The zero-dimensional case is $B = \emptyset$, *i.e.*, $X = \{0\}$. Our first step is to study $\mathbf{C}^n$, the standard n-dimensional vector space, given a positive integer n.

v Finite-dimensional spaces

The product topology of $\mathbf{C}^n$

The space $\mathbf{C}^n$, the n-th power of $\mathbf{C}$, is seen as the set of all lists

$$(z_1, \dots, z_n) = z_1 \cdot (\underbrace{1, 0, \dots}_{1_{\{1\}}}) + \cdots + z_n \cdot (\underbrace{0, \dots, 1}_{1_{\{n\}}}), \tag{16}$$

where each variable $z_1, \dots, z_n$ ranges over $\mathbf{C}$. It is easily shown that $1_{\{1\}}, \dots, 1_{\{n\}}$ enumerates a basis of $\mathbf{C}^n$, known as the *standard* basis. A common notation is $e_1 = 1_{\{1\}}, \dots, e_n = 1_{\{n\}}$. Furthermore, $\mathbf{C}^n$ is endowed with the product topology generated by the origin-centered polydiscs

$$\{(z_1, \dots, z_n) : |z_1| < r_1, \dots, |z_n| < r_n\} \qquad (\min(r_1, \dots, r_n) > 0) \tag{17}$$

and their translates over $\mathbf{C}^n$. Next, we state that every n-dimensional topological vector space is linearly homeomorphic to $\mathbf{C}^n$.

Hausdorff topology of a finite-dimensional vector space

By "vector topology" or "topological vector space", we will always refer to the Hausdorff topology for vector spaces, as defined in [1.6, 1.12] of [4]. In this context, let Y be a vector space of finite dimension n. A striking consequence of Theorem [1.21] of [4] is that

> Y *has exactly one vector topology, which is a linearly homeomorphic image of the topology of* $\mathbf{C}^n$.

Clearly, Y is locally compact as well. Moreover, it is locally convex and locally bounded. Therefore, by [1.39] of [4], Y is normable. A fundamental result is that any two norms $\|\cdot\|$ and $\|\cdot\|'$ on Y are *equivalent*, in the sense that there exist two positive constants C and C′ such that, for all $y \in Y$,

$$\|y\|' \leq C\,\|y\|, \tag{18}$$

$$\|y\| \leq C'\|y\|'. \tag{19}$$

The reader will find all proofs in [A.1]. The following provides all optimal C for the standard norms 1, 2, and ∞.

The standard norms 1, 2, and ∞

When $\mathbf{C}^n$ is equipped with the standard norms $i, j = 1, 2, \infty$, the smallest possible constants $C = C_{i,j}$ such that

$$\|z\|_j \leq C\|z\|_i \qquad \big(z \in \mathbf{C}^n\big) \tag{20}$$

are easily derived from the definitions of these norms, except for the case $C_{2,1} = \sqrt{n}$, which requires the Cauchy–Schwarz inequality; see [12.2] of [4]. The table below records these optimal $C_{i,j}$:

i \ j	1	2	∞
1	1	1	1
2	$\sqrt{n}$	1	1
∞	n	$\sqrt{n}$	1

Table 1. Optimal $C_{i,j}$ for the norms $i, j = 1, 2, \infty$.

vi Continuity and boundedness in normed spaces

A linear mapping $\Lambda : X \to Y$ is said to be *bounded* **iff** $\Lambda(E)$ *is bounded when* $E \subseteq X$ *is bounded*; see [1.31] of [4]. Given normed spaces X and Y, bounded linear maps $\Lambda : X \to Y$ form a vector space $B(X, Y)$. Indeed, in this context, linearity implies that Λ is bounded **iff** Λ is bounded on the closed unit ball of X. This condition reads as

$$\sup_{\|x\| \leq 1} \|\Lambda x\| < \infty. \tag{21}$$

In this case, the supremum is called *the operator norm* of Λ and is denoted by $\|\Lambda\|$. The map $\Lambda \mapsto \|\Lambda\|$ makes $B(X, Y)$ a normed space; see [4.1] of [4]. Furthermore, [1.32] of [4] shows that, in the current context, boundedness coincides with continuity. When Y is the scalar field, this equivalence also comes from [1.18] of [4]. The property of equicontinuity of a collection $\Gamma \subseteq B(X, Y)$ now reads as

$$\sup_{\Lambda \in \Gamma} \|\Lambda\| < \infty. \tag{22}$$

Thus, *uniform boundedness* and equicontinuity coincide as well; *cf.*[2.4] of [4].

IV Radon measure theory

i Radon measures

Given a locally compact Hausdorff space X, *e.g.*, $\mathbf{R}^n$, a positive Radon measure $\Lambda : C_c(X) \to \mathbf{C}$ is a linear functional that is *positive*, in the sense that

$$\Lambda(\phi) \geq 0 \qquad (\phi \in C_c(X) : \phi \geq 0). \tag{23}$$

Theorem [2.14] of [3] shows that positivity implies the following *continuity* property: to every compact $K \subseteq X$ corresponds a constant $M_K < \infty$ such that

$$|\Lambda(\phi)| \leq M_K \|\phi\|_\infty \qquad (\phi \in C_c(X) : \operatorname{supp}(\phi) \subseteq K). \tag{24}$$

Note that some authors take this property as the definition of a Radon measure, in a more general sense; see [5]. Furthermore, Λ is bounded on $C_c(X)$ with respect to the supremum norm **iff** all M_K can be chosen with a common bound M ($M_K \leq M < \infty$). In this case, the functionals Λ are continuous in the sense that they satisfy (21); they precisely constitute the space of norm-bounded Radon measures. By density of $C_c(X)$ in $C_0(X)$, each of them extends uniquely to $C_0(X)$; according to [6.19] of [3], it can then be isometrically identified with a specific *complex* regular Borel measure μ. The isometry $\Lambda \mapsto \mu$ reads as

$$\|\Lambda\| = |\mu|(X) \triangleq \sup\left\{\sum_{i=1}^{\infty} |\mu(E_i)| : E_i \text{ measurable, } \{E_1, E_2, \dots\} \text{ partition of } X\right\}, \tag{25}$$

where the quantity $|\mu|(X)$ is called the *total variation* of μ (with respect to X). An interesting special case is that of X compact, which makes every Radon measure norm-bounded. This simpler version is addressed in Exercise [2.3]. Note that, in measure theory, by *the support* of μ, we mean:

$$\operatorname{supp}(\mu) \triangleq X \setminus \bigcup\{V \text{ open} : |\mu|(V) = 0\}. \tag{26}$$

Obviously, this definition transfers to positive measures ν, where the condition $|\nu|(V) = 0$ reads simply $\nu(V) = 0$. A very important Radon measure, the *Dirac measure* δ, is described in [A.3]. Identified with a Borel measure (also denoted by δ), δ has range $\{0, 1\}$ and support $\{0\}$.

ii Lebesgue integration

When X is also σ-compact, Theorems [2.14, 2.18] of [3] together establish that every positive Radon measure Λ is identified with a regular positive Borel measure β over X. By "regular", we mean that, for each Borel set E,

$$\beta(E) = \sup\{\beta(K) : K \subseteq E, K \text{ compact}\} = \inf\{\beta(U) : E \subseteq U, U \text{ open}\}. \tag{27}$$

More specifically, in terms of the Lebesgue integral, Λ satisfies

$$\Lambda(\phi) = \int_X \phi \, d\beta \qquad \big(\phi \in C_c(X)\big), \tag{28}$$

where, given an open set U, $\beta(U)$ is the integral

$$\int_U d\beta = \int_X 1_U \, d\beta = \sup\{\Lambda(\phi) : \operatorname{supp}(\phi) \subseteq U, 0 \leq \phi \leq 1\}. \tag{29}$$

Integration of indicators 1_U is generalized to bounded measurable functions f (more generally, to locally integrable f, see the RHS of (33)) in the sense that

$$\Lambda_f(\phi) \triangleq \int_X \phi \, f \, d\beta \tag{30}$$

constructs a Radon measure $\Lambda_f : C_c(X) \to \mathbf{C}$ in the sense of [5]. Observe that f and the linear functional Λ_f are identified modulo $\mathcal{N}_\beta = \{\omega : \omega \text{ measurable}, \omega = 0 \ \beta\text{-a.e.}\}$ in (30). In terms of cosets, this reads

$$\Lambda_g = \Lambda_f \iff g - f \in \mathcal{N}_\beta. \tag{31}$$

A central question in calculus is whether the integral of the modulus, namely

$$|f|_1 \triangleq \int_X |f| \, d\beta, \tag{32}$$

is finite. This motivates the following definitions, valid for measurable functions f:

$$L^1(\beta) \triangleq \{f : |f|_1 < \infty\} \subseteq L^1_{\text{loc}}(\beta) \triangleq \{f : |1_K f|_1 < \infty \text{ for every compact } K\}. \tag{33}$$

The standard notation L^1 is used when the context is clear. Additionally, in the context of integration on $\mathbf{R}^n$, β denotes the usual Borel measure, that is, the Lebesgue measure restricted to Borel sets.

V L^p spaces

From now on, X is, more generally, a measurable space equipped with a positive measure ν. The previous definitions of spaces $\mathcal{N}_\beta$, $L^1(\beta)$ and magnitude sum $|\cdot|_1$ extend immediately. A key point is that $L^1 = L^1(\nu)$, equipped with $|\cdot|_1$, is still a seminormed space. These definitions and properties are generalized to $1 \leq p < \infty$, as follows:

$$|f|_p \triangleq \left[\int_X |f|^p \, d\nu\right]^{1/p} \qquad (f \text{ measurable}), \tag{34}$$

$$L^p \triangleq \{f : |f|_p < \infty\}, \tag{35}$$

$$\mathbf{L}^p \triangleq L^p/\mathcal{N}_\nu, \tag{36}$$

$$\|f + \mathcal{N}_\nu\|_p \triangleq |f|_p \qquad (f \in L^p). \tag{37}$$

The subadditivity property

$$|f + g|_p \leq |f|_p + |g|_p \tag{38}$$

is known as *Minkowski's inequality*. Combined with the obvious homogeneity $|\alpha f|_p = |\alpha|\,|f|_p$, Minkowski's inequality shows that $|\cdot|_p$ is a seminorm on L^p and that $\|\cdot\|_p$ is a norm on $\mathbf{L}^p$. Define

$$|f|_\infty \triangleq \inf\{M : |f| \leq M \ \nu\text{-a.e.}\} \qquad (\text{convention: } \inf\emptyset = \infty) \tag{39}$$

for all measurable functions f. This extends the notions of $\mathbf{L}^p$ space and $\|\cdot\|_p$ to $p = \infty$, as follows:

$$L^\infty \triangleq \{f \text{ measurable} : |f|_\infty < \infty\}, \tag{40}$$

$$\mathbf{L}^\infty \triangleq L^\infty/\mathcal{N}_\nu, \tag{41}$$

$$\|f + \mathcal{N}_\nu\|_\infty \triangleq |f|_\infty \qquad (f \in L^\infty). \tag{42}$$

A crucial point is that every $\mathbf{L}^p$, equipped with $\|\cdot\|_p$, is a Banach space; see Theorem [3.11] of [3].

Chapter 1

Topological Vector Spaces

1 Exercise 1. Basic results

Suppose X *is a vector space. All sets mentioned below are understood to be subsets of* X. *Prove the following statements from the axioms as given in section 1.4.*

(a) If $x, y \in X$ *there is a unique* $z \in X$ *such that* $x + z = y$.

(b) $0 \cdot x = 0 = \alpha \cdot 0 \qquad (\alpha \in \mathbf{C}, x \in X)$.

(c) $2A \subseteq A + A$.

(d) A *is convex if and only if* $(s + t)A = sA + tA$ *for all positive scalars* s *and* t.

(e) Every union (and intersection) of balanced sets is balanced.

(f) Every intersection of convex sets is convex.

(g) If Γ *is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of* Γ *is convex.*

(h) If A *and* B *are convex, so is* $A + B$.

(i) If A *and* B *are balanced, so is* $A + B$.

(j) Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.

PROOF.

(a) Such a property only depends on the group structure of X: Each x in X has an additive inverse -x. Let x' be any additive inverse of x, so that $x - x = 0 = x + x'$. Thus, $-x + x - x = -x + x + x'$, which is equivalent to $-x = x'$. Therefore, the inverse -x is unique. It is now clear that $x + z = y$ **iff** $z = -x + y$, which asserts both the existence and the uniqueness of z.

(b) Remark that

$$0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x \tag{1.1}$$
$$= (0 + 0) \cdot x = 0 + 0 \cdot x \tag{1.2}$$

then conclude from (a) that $0 \cdot x = 0$. So,

$$0 = 0 \cdot x = (1 - 1) \cdot x = x + (-1) \cdot x \Rightarrow -1 \cdot x = -x. \tag{1.3}$$

Finally,

$$\alpha \cdot 0 \overset{(1.3)}{=} \alpha \cdot (x + (-1 \cdot x)) = \alpha \cdot x + \alpha \cdot (-1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0, \tag{1.4}$$

which proves (b).

(c) Remark that

$$2x = (1+1)x = x + x \tag{1.5}$$

for every x in X, and so conclude that

$$2A = \{2x : x \in A\} = \{x + x : x \in A\} \subseteq \{x + y : (x, y) \in A^2\} = A + A \tag{1.6}$$

for all subsets A of X; which proves (c).

(d) If A is convex, then

$$A \subseteq \frac{s}{s+t}A + \frac{t}{s+t}A \subseteq A; \tag{1.7}$$

which is

$$sA + tA = (s+t)A. \tag{1.8}$$

Conversely, the special case $s + t = 1$ is

$$sA + (1-s)A = A. \tag{1.9}$$

The latter extends to $s = 0$, since

$$0A + A \overset{(b)}{=} \{0\} + A = A. \tag{1.10}$$

The extension to $s = 1$ is analogously established (or simply use the fact that $+$ is commutative!). So ends the proof.

(e) Let A range over a collection Γ of balanced subsets, so that

$$\alpha \bigcap \Gamma \subseteq \alpha A \subseteq A \subseteq \bigcup \Gamma \tag{1.11}$$

for all scalars α of magnitude ≤ 1. The inclusion $\alpha \bigcap \Gamma \subseteq A$ establishes the first part. Now remark that

$$\alpha A \subseteq \bigcup \Gamma \tag{1.12}$$

implies

$$\alpha \bigcup \Gamma \subseteq \bigcup \Gamma; \tag{1.13}$$

which completes the proof.

(f) Let A range over Γ, a collection of convex subsets, so that

$$(s+t)\bigcap \Gamma \subseteq s\bigcap \Gamma + t\bigcap \Gamma \subseteq sA + tA \overset{(d)}{\subseteq} (s+t)A \tag{1.14}$$

for all positives scalars s and t. Inclusions at both extremities force

$$s\bigcap \Gamma + t\bigcap \Gamma = (s+t)\bigcap \Gamma. \tag{1.15}$$

We then conclude from (d) that the intersection of C is convex. So ends the proof.

(g) We dismiss all trivial cases $\Gamma = \emptyset$, $\{\emptyset\}$, $\{\{x\}\}$, $\{\emptyset, \{x\}\}$ then pick x_1, x_2 in $\bigcup \Gamma$, so that each x_i $(i = 1, 2)$ lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2. So, x_1, x_2 are now elements of the convex set C_2. Every convex combination of our x_i's is then in $C_2 \subseteq \bigcup \Gamma$. Hence (g).

(h) Simply remark that

$$s(A+B) + t(A+B) = sA + tA + sB + tB = (s+t)(A+B) \tag{1.16}$$

for all positive scalars s and t, then conclude from (d) that $A + B$ is convex.

(i) Given any α from the closed unit disc,

$$\alpha(A + B) = \alpha A + \alpha B \subseteq A + B. \tag{1.17}$$

This completes the proof: $A + B$ is balanced.

(j) The proof is based on Lemma [A.1]. Let Γ be a collection of vector subspaces of X. Define $I = \bigcap \Gamma$ and $U = \bigcup \Gamma$. The intersection I is convex and balanced by (e) and (f), because every $Y \in \Gamma$ is convex and balanced. Next, observe that

$$I + I \subseteq Y + Y \subseteq Y \tag{1.18}$$

for all $Y \in \Gamma$. Thus,

$$I + I \subseteq I. \tag{1.19}$$

It now follows from the implication (b) $\Rightarrow$ (a) of Lemma [A.1] that I is a vector subspace of X. We now prove the counterpart of (g) when Γ is totally ordered by set inclusion. Combining (e) with (g) demonstrates that U is convex and balanced. To show that U is a vector subspace, we note that this total ordering of Γ implies

$$Y_1 + Y_2 \subseteq \max(Y_1, Y_2), \tag{1.20}$$

where Y_1 and Y_2 run through Γ. Hence

$$U + U \subseteq U. \tag{1.21}$$

It then follows from the implication (b) $\Rightarrow$ (a) of Lemma [A.1] that U is a vector subspace of X. To prove the counterpart of (h), let each Y_i be a vector subspace of X. Taken together, (h) and (i) imply that $Y_1 + Y_2$ is convex and balanced. Moreover,

$$(Y_1 + Y_2) + (Y_1 + Y_2) = (Y_1 + Y_1) + (Y_2 + Y_2) \subseteq Y_1 + Y_2. \tag{1.22}$$

Finally, we conclude from (b) $\Rightarrow$ (a) of Lemma [A.1] that $Y_1 + Y_2$ is a vector subspace of X.

□

2 Exercise 2. Convex hull

The convex hull $\mathrm{co}(A)$ *of a set A in a vector space X is the set of all convex combinations of members of A, that is, the set of all sums* $t_1x_1 + \cdots + t_nx_n$ *in which* $x_i \in A$, $t_i \geq 0$, $\sum t_i = 1$*; n is arbitrary. Prove that the convex hull of a set A is convex and that it is the intersection of all convex sets that contain A.*

PROOF. By definition, a term t_ix_i with $t_i = 0$ makes no contribution to the sum $t_1x_1 + \cdots + t_nx_n$. We henceforth assume without loss of generality that all t_i are positive. The case $n = 1$ shows that $A \subseteq \mathrm{co}(A)$. Given $0 \leq s \leq 1$, the linear combination

$$s\sum_{i=1}^{n} t_ix_i + (1-s)\sum_{j=1}^{n'} t'_jx'_j = \sum_{i=1}^{n} st_ix_i + \sum_{j=1}^{n'}(1-s)t'_jx'_j, \tag{1.23}$$

when both $\sum t_ix_i$ and $\sum t'_jx'_j$ range over co(A), is also a convex combination of $\{x_1, \ldots, x_n, x'_1, \ldots, x'_{n'}\}$. To see this, observe that the sum of the *weights* st_i and $(1-s)t'_j$ equals 1. In detail,

$$\sum_{i=1}^{n} st_i + \sum_{j=1}^{n'}(1-s)t'_j = s\sum_{i=1}^{n} t_i + (1-s)\sum_{j=1}^{n'} t'_j = 1. \tag{1.24}$$

This proves that

$$s\,\mathrm{co}(A) + (1-s)\,\mathrm{co}(A) \subseteq \mathrm{co}(A) \tag{1.25}$$

for all $0 \leq s \leq 1$. In other words, co(A) is convex.

To prove the intersection property, we establish that

$$\text{A is convex} \iff \text{co}(\text{A}) = \text{A}. \tag{1.26}$$

The direction $\Leftarrow$ follows directly from the preceding part, which establishes the convexity of co(A). We now address the $\Rightarrow$ direction, which we actually need for the proof. Suppose that A is convex. An induction on n proves that

$$t_1\text{A} + \cdots + t_n\text{A} = (t_1 + \cdots + t_n)\,\text{A} \tag{1.27}$$

for all positive t_i, as (1.8) gives the base case $n = 2$. The detail is left to the reader. In particular, $t_1\text{A} + \cdots + t_n\text{A} = \text{A}$ when $t_1 + \cdots + t_n = 1$. In that case, the sums $t_1\text{A} + \cdots + t_n\text{A} = \text{A}$ of (1.27) together cover exactly co(A). Taking the union of both sides of (1.27) over all n and all such t_i gives $\text{co}(\text{A}) = \text{A}$, which establishes (1.26). We now return to the intersection property. To this end, we define

$$\Gamma \triangleq \big\{\text{B} : \text{B is convex and contains A}\big\}. \tag{1.28}$$

Our goal then translates into proving that

$$\text{co}(\text{A}) = \bigcap \Gamma. \tag{1.29}$$

First, it follows from the preceding that $\text{co}(\text{A}) \in \Gamma$, by the very definition of Γ. So,

$$\bigcap \Gamma \subseteq \text{co}(\text{A}). \tag{1.30}$$

To obtain the converse inclusion, we apply (1.26) to each member B of Γ in the role of A. Then,

$$\text{co}(\text{A}) \subseteq \text{co}(\text{B}) = \text{B} \qquad (\text{B} \in \Gamma), \tag{1.31}$$

as $\text{A} \subseteq \text{B}$ implies $\text{co}(\text{A}) \subseteq \text{co}(\text{B})$. Equivalently,

$$\text{co}(\text{A}) \subseteq \bigcap \Gamma. \tag{1.32}$$

Finally, combining (1.30) with (1.32) gives (1.29). □

3 Exercise 3. Other basic results

Let be X as topological vector space. All sets mentioned below are understood to be the subsets of X. Prove the following statements:

(a) The convex hull of every open set is open.

(b) If X is locally convex then the convex hull of every bounded set is bounded.

(c) If A and B are bounded, so is A+B.

(d) If A and B are compact, so is A+B.

(e) If A is compact and B is closed, then A+B is closed.

(f) The sum of two closed sets may fail to be closed.

PROOF. **Proof of (a).** Let A be open, so that

$$\text{co}(\text{A}) = \bigcup \big\{\text{co}(\text{V}_1, \dots, \text{V}_n) : \text{V}_i \subseteq \text{A},\ \text{V}_i \text{ open, n is arbitrary}\big\}, \tag{1.33}$$

where $\text{co}(\text{V}_1, \dots, \text{V}_n)$ is the union of all convex combinations $t_1\text{V}_1 + \cdots + t_n\text{V}_n$ ($t_1 + \cdots + t_n = 1$, $t_i > 0$). A straightforward induction on n proves that every mapping $(x_1, \dots, x_n) \mapsto t_1x_1 + \cdots + t_nx_n$ is open. The base cases $n = 1, 2$ are provided by [1.7] of [4]. As a result, every $t_1\text{V}_1 + \cdots + t_n\text{V}_n$ is an open subset of $\text{co}(\text{V}_1, \dots, \text{V}_n)$. Therefore, $\text{co}(\text{V}_1, \dots, \text{V}_n)$ is open. Property (a) follows, since co(A) is the union of all $\text{co}(\text{V}_1, \dots, \text{V}_n)$.

Proof of (b). Given a bounded E, let V be any neighborhood of 0. By [1.14 (b)] of [4], V contains a convex neighborhood W of 0. It follows that there exists $s > 0$ such that

$$E \subseteq tW \subseteq tV \qquad (t > s). \tag{1.34}$$

Hence

$$\operatorname{co}(E) \subseteq \operatorname{co}(tW) = t \operatorname{co}(W) = tW \subseteq tV, \tag{1.35}$$

which completes the proof.

Proof of (c). For a fixed neighborhood V of the origin, the continuity of $+$, combined with [1.14] of [4], provides a balanced open U such that

$$U + U \subseteq V. \tag{1.36}$$

By boundedness, $A \subseteq rU$ for some $r > 0$, and $B \subseteq sU$ for some $s > 0$. For $t > \max(r, s)$, the balancedness of U gives

$$A + B \subseteq rU + sU \tag{1.37}$$
$$\subseteq tU + tU \tag{1.38}$$
$$\subseteq tV, \qquad \big(\text{from (1.36)}\big) \tag{1.39}$$

which concludes the proof.

Proof of (d). Since $A \times B$ is compact and the mapping

$$\begin{aligned} + : A \times B &\to A + B \\ (a, b) &\mapsto a + b \end{aligned} \tag{1.40}$$

is continuous and surjective, the sum $A + B$ is compact.

Proof of (e). Assume that $A \neq \emptyset$ and $B \neq \emptyset$, since otherwise the result is trivial. Pick $c \in X \setminus (A + B)$. It suffices to show that $c \in X \setminus \overline{A + B}$. To this end, let a range over A. The set $a + B$ is closed; see [1.7] of [4]. Since $(a + B) \cap \{c\} = \emptyset$, it follows from [1.10] of [4] that there exists a neighborhood $V = V(a)$ of the origin such that

$$(a + B + V) \cap (c + V) = \emptyset. \tag{1.41}$$

As a compact subset, A is covered by finitely many $a + V$, say $a_1 + V_1, \ldots, a_N + V_N$, whose union is U. Hence

$$A + B \subseteq U + B. \tag{1.42}$$

Now choose

$$W \triangleq \bigcap_{i=1}^{N} V_i \tag{1.43}$$

so that

$$(a_i + V_i + B) \cap (c + W) \overset{(1.41)}{=} \emptyset \tag{1.44}$$

for every $a_i + V_i$. Thus, $c \notin \overline{U + B}$. Finally, (1.42) establishes that $c \notin \overline{A + B}$. This completes the proof.

Proof of (f). We provide a counterexample. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous such that

(i) $f(x) + f(-x) \neq 0 \qquad (x \in \mathbf{R})$,

(ii) f vanishes at infinity.

Condition (i) is satisfied when $f(x) > 0$ for all x. For simplicity, choose f to be strictly positive and even, *e.g.*, the *bell curve* $f(x) = \exp(-x^2)$. Other canonical choices are $f(x) = 2^{-|x|}$, $f(x) = \frac{1}{1+|x|}$, and so on. As a continuous function, f has a closed graph G; see [2.14] of [4]. Observe that $(0,0) \notin G + G$ because$(0,0) \in G + G$ implies

$$(0,0) = \Big(x_0 + (-x_0),\ \underbrace{f(x_0) + f(-x_0)}_{>0}\Big) \tag{1.45}$$

for some x_0, which contradicts (i). However,

$$\Big(0, f(n) + f(-n)\Big) \in G + G \qquad (n = 1, 2, 3, \dots). \tag{1.46}$$

Finally, it follows from (ii) that

$$\Big(0, f(n) + f(-n)\Big) \xrightarrow[n\to\infty]{} (0,0). \tag{1.47}$$

We have thus constructed a sequence in $G + G$ that converges to $(0,0) \notin G + G$. The proof is complete.

□

Lemma 1.1 Corollary to (e).

If $B = \overline{S}$, then

$$A + B \subseteq \overline{A + S} \subseteq \overline{A + B} = A + B \tag{1.48}$$

The special case where $A = \{x\}$ and $B = X$ applies to Exercise 15 in chapter 2.

PROOF. The result directly follows from [1.13 (b)] of [4], since A is closed; see [1.12] of [4]. □

4 Exercise 4. A balanced set whose interior is not balanced

Let $B = \big\{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \le |z_2|\big\}$. *Show that* B *is balanced but that its interior is not. [compare with (e) of Theorem 1.13.]*

PROOF. Let us equip $\mathbf{C}^2$ with the 1-norm, that is, $\|(z_1, z_2)\|_1 = |z_1| + |z_2|$ for every pair $(z_1, z_2) \in \mathbf{C}^2$. Note that B is nonempty, since the origin $(0,0)$ lies in B. The proof consists of establishing that

(a) B is balanced,

(b) B has nonempty interior,

(c) the interior B° does not contain the origin.

This suffices, since every nonempty balanced subset contains the origin. Observe that

$$|\alpha \cdot z_1| = |\alpha| \cdot |z_1| \le |\alpha| \cdot |z_2| = |\alpha \cdot z_2| \qquad \big((z_1, z_2) \in B, \alpha \in \mathbf{C}\big). \tag{1.49}$$

In particular, this holds when $|\alpha| \le 1$. Therefore, B is balanced. This is (a). The triangle inequality gives

$$\big\{(z_1, z_2) : \|(1 - z_1, 2 - z_2)\|_1 < 1/3\big\} \subseteq B. \tag{1.50}$$

Hence $B^\circ \neq \emptyset$, which is (b). We now prove (c). To this end, suppose for the sake of contradiction that B contains the origin-centered open ball

$$U_r \triangleq \big\{(z_1, z_2) : \|(z_1, z_2)\|_1 < r\big\} \tag{1.51}$$

for some $r > 0$. The special case $(r/2, 0) \in U_r$ conflicts with the definition of B. This completes the proof. □

Remark. This result implies that the assumption $0 \in B^\circ$ cannot be omitted from [1.13 (e)] of [4].

5 Exercise 5. A first restatement of boundedness

Consider the definition of "bounded set" given in Section 1.6. Would the content of this definition be altered if it merely required that to every neighborhood V *of 0 corresponds* some $t > 0$ *such that* $E \subseteq tV$*?*

PROOF. The answer is: No. To prove this, start from (a) of Section 1.14: V contains W, a balanced neighborhood of 0. Assume that E is bounded in this weaker sense, *i.e.*, there exists a positive t that satisfies

$$E \subseteq tW. \tag{1.52}$$

Hence

$$E \subseteq tW \subseteq sW \subseteq sV \qquad (s > t) \tag{1.53}$$

because W is balanced. We then recover the definition given in Section 1.6: The two definitions are equivalent. □

6 Exercise 6. A second restatement of boundedness

Prove that a set E *in a topological vector space is bounded if and only if every countable subset of E is bounded.*

PROOF. It is clear that every subset of a bounded set is bounded. Conversely, assume that E is not bounded then pick V a neighborhood of the origin: No integer $n = 1, 2, \dots$ satisfies $E \subseteq nV$ (see Exercise 1 in Chapter 1). In other words, there exists a sequence $\{x_1, \dots, x_n, \dots\} \subseteq E$ such that

$$x_n \notin nV. \tag{1.54}$$

As a consequence, x_n/n fails to converge to 0 as n tends to ∞. In contrast, $1/n$ succeeds. It then follows from Section 1.30 that $\{x_1, \dots, x_n, \dots\}$ is not bounded. So ends the proof. □

7 Exercise 7. Metrizability and number theory

Let X be the vector space of all complex functions on the unit interval $[0, 1]$*, topologized by the family of seminorms*

$$p_x(f) = |f(x)| \qquad (0 \le x \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology.

Show that there is a sequence $\{f_n\}$ *in X such that (a)* $\{f_n\}$ *converges to* 0 *as* $n \to \infty$*, but (b) if* $\{\gamma_n\}$ *is any sequence of scalars such that* $\gamma_n \to \infty$ *then* $\{\gamma_n f_n\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to* 0 *has the same cardinality as* $[0, 1]$*.) This shows that metrizability cannot be omitted in (b) of Theorem 1.28.*

PROOF.

Justifying the terminology. The subsets

$$V(x, k) \triangleq \{p_x < 1/k\} \qquad (x \in [0, 1], k = 1, 2, 3, \dots) \tag{1.55}$$

of X form a local *subbase* of a vector space topology τ. A local base $\mathcal{B}$ of τ consists of all finite intersections of these $V(x, k)$; see Theorem [1.37] of [4]. In this part, we prove that τ-convergence is exactly pointwise convergence in X. To this end, given

(i) any $f \in X$,

(ii) a sequence of functions f_n in X, and

(iii) finitely many subsets $S_1, \dots, S_j, \dots, S_N$ of X, with union T,

we equip X with the counting measure

$$\Delta \triangleq \sum_{n=1}^{\infty} \delta_{f_n - f}. \tag{1.56}$$

In terms of Iverson brackets, this reads as

$$\int_S d\Delta = \sum_{n=1}^{\infty} \big[f_n - f \in S \big] \qquad (S \subseteq X). \tag{1.57}$$

Note that the subadditivity and the monotonicity of Δ give

$$\int_T d\Delta \leq \sum_{j=1}^{N} \int_{S_j} d\Delta \leq N \int_T d\Delta. \tag{1.58}$$

First, assume that the f_n τ-converge to f. Since every $V(x,k)$ is a neighborhood of 0,

$$\int_{X \setminus V(x,k)} d\Delta = \sum_{n=1}^{\infty} \big[\underbrace{f_n - f \in X \setminus V(x,k)}_{|f_n(x) - f(x)| \geq 1/k} \big] \in \mathbf{N} \tag{1.59}$$

for all $V(x,k)$. Thus, for every x and every k, the inequality

$$\big| f_n(x) - f(x) \big| \geq 1/k \tag{1.60}$$

holds for *only* finitely many n. Equivalently,

$$\big| f_n(x) - f(x) \big| < 1/k \tag{1.61}$$

for *all* but finitely many n. As k is arbitrary, this establishes that, for every $x \in [0,1]$,

$$f_n(x) \xrightarrow[n \to \infty]{} f(x). \tag{1.62}$$

We have just proved that, in X, τ-convergence implies pointwise convergence.

Conversely, fix the function f and assume that our f_n do not τ-converge to f. Consequently, there exists $W \in \mathcal{B}$ such that infinitely many $f_n - f$ are *outside* W. Equivalently,

$$\int_{X \setminus W} d\Delta = \infty. \tag{1.63}$$

As W is of the form $\bigcap_{j=1}^{N} V(x_j, k_j)$, we choose, in (iii),

$$\begin{aligned} S_j &\triangleq \big\{ g : \big| g(x_j) \big| \geq 1/k_j \big\} \qquad (j = 1, \dots, N), \\ T &\triangleq \bigcup_{j=1}^{N} S_j = X \setminus W, \end{aligned} \tag{1.64}$$

so that

$$\int_T d\Delta = \int_{X \setminus W} d\Delta = \infty. \tag{1.65}$$

By (1.58), we thus obtain

$$\sum_{j=1}^{N} \int_{S_j} d\Delta = \infty. \tag{1.66}$$

Because this sum has only N terms, there must be an index j such that

$$\int_{S_j} d\Delta = \infty. \tag{1.67}$$

In other words,

$$\left| f_n(x_j) - f(x_j) \right| \geq 1/k_j \tag{1.68}$$

for infinitely many n. Thus, the sequence $\left(f_1(x_j), \dots, f_n(x_j), \dots\right)$ fails to converge to $f(x_j)$; *cf.* (1.61). This completes the proof that τ-convergence coincides with pointwise convergence on X. We prove the second part in two different ways. The first proof uses the suggested cardinality argument, while the second one relies only on binary expansions.

> *From now on, we assume without loss of generality that all* γ_n *are positive; the general case is covered by substituting* $\max(1, |\gamma_n|)$ *for each original scalar* γ_n.

Proof with the given hint. We prove the second part by constructing a specific sequence of functions $f_n \in X$ that satisfies both (a) and (b). Following the hint, we assume a bijection

$$\begin{aligned} \phi : \mathbf{C}^{\mathbf{N}_+} \cap \left\{\theta : \lim_{\infty} \theta = 0\right\} &\to [0,1] \\ (\theta_1, \dots, \theta_n, \dots) &\mapsto x. \end{aligned} \tag{1.69}$$

For each $x \in [0,1]$, define

$$f_n(x) \triangleq \theta_n \tag{1.70}$$

with the convention that $(\theta_1, \dots, \theta_n, \dots) = \phi^{-1}(x)$. Consequently, the sequence $(f_1, \dots, f_n, \dots)$ tends pointwise to 0. By the first part, this is (a). With this construction, given a positive sequence $\gamma = (\gamma_1, \dots, \gamma_n, \dots)$ that tends to ∞, the sequence θ_γ with general term

$$\theta_{\gamma,n} \triangleq \frac{1}{\sqrt{\gamma_n}} \tag{1.71}$$

tends to 0. The input

$$x_\gamma \triangleq \phi(\theta_\gamma) \tag{1.72}$$

is then well-defined. Hence

$$\gamma_n f_n(x_\gamma) \overset{(1.70)}{=} \gamma_n \theta_{\gamma,n} \tag{1.73}$$

$$\overset{(1.71)}{=} \sqrt{\gamma_n} \xrightarrow[n\to\infty]{} \infty. \tag{1.74}$$

In conclusion, $\left(\gamma_1 f_1(x_\gamma), \dots, \gamma_n f_n(x_\gamma), \dots\right)$ diverges for every sequence $\gamma_n \to \infty$; by the first part, this proves (b). We now construct another sequence $(f_1, \dots, f_n, \dots)$ without the hint.

Proof with binary expansions (no hint). This argument is based on the following property: a real number is irrational **iff** its binary expansion is not eventually periodic. More precisely, there exists a bijection

$$\begin{aligned} \psi : \{0,1\}^{\mathbf{N}_+} \cap \left\{\beta : \beta \text{ is not eventually periodic}\right\} &\to [0,1] \setminus \mathbf{Q} \\ (\beta_1, \dots, \beta_n, \dots) &\mapsto \sum_{k=1}^{\infty} \beta_k 2^{-k}. \end{aligned} \tag{1.75}$$

For each $x \in [0,1]$, define

$$f_n(x) \triangleq [x \notin \mathbf{Q}] \cdot \frac{1}{2^{\beta_1 + \cdots + \beta_n}} \tag{1.76}$$

with the convention that $x = \psi(\beta)$ whenever x is irrational. This bit stream is not eventually periodic and therefore has infinitely many 1s, which implies that $\beta_1 + \cdots + \beta_n$ goes to infinity as $n \to \infty$. Thus, for all x,

$$f_n(x) \xrightarrow[n\to\infty]{} 0. \tag{1.77}$$

By the first part, (1.77) is (a). Next, let $\gamma_n \to \infty$. For each positive integer k, there exists a threshold m_k beyond which $\gamma_n > 3^k$. Fixing $n_0 = 0$, we recursively select $n_k > m_k$ such that

$$n_k - n_{k-1} \geq k. \tag{1.78}$$

The core idea of the construction is that

$$n_k - n_{k-1} \xrightarrow[k\to\infty]{} \infty. \tag{1.79}$$

Indeed, this turns the indicator function of $\{n_1, n_2, \dots\}$ on $\mathbf{N}_+$, that is,

$$\begin{aligned} \beta_\gamma : \mathbf{N}_+ &\to \{0,1\} \\ n &\mapsto [\,n \text{ is one of these integers } n_k\,], \end{aligned} \tag{1.80}$$

into a sequence that is *not eventually periodic*. Consequently, the input

$$x_\gamma \triangleq \psi(\beta_\gamma) \tag{1.81}$$

is irrational. Thus, by (1.76),

$$f_{n_k}(x_\gamma) = \frac{1}{2^{\beta_{\gamma,1}+\cdots+\beta_{\gamma,n_1}+\cdots+\beta_{\gamma,n_k}}}. \tag{1.82}$$

Moreover, $\beta_{\gamma,n} = 1$ **iff** $n \in \{n_1, n_2, \dots\}$, which yields

$$\beta_{\gamma,1} + \cdots + \beta_{\gamma,n_1} + \cdots + \beta_{\gamma,n_k} = k. \tag{1.83}$$

Combining (1.82) with (1.83), we obtain, since $\gamma_{n_k} > 3^k$,

$$\gamma_{n_k} f_{n_k}(x_\gamma) > \left(\frac{3}{2}\right)^k \xrightarrow[k\to\infty]{} \infty. \tag{1.84}$$

In conclusion, every $\gamma_n \to \infty$ contains a subsequence $(\gamma_{n_1}, \dots, \gamma_{n_k}, \dots)$ such that $(\gamma_{n_1} f_{n_1}, \dots, \gamma_{n_k} f_{n_k}, \dots)$ diverges; by the first part, this proves (b).

□

9 Exercise 9. Quotient map

Suppose

(a) X *and* Y *are topological vector spaces,*

(b) $\Lambda : X \to Y$ *is linear.*

(c) N *is a closed subspace of* X*,*

(d) $\pi : X \to X/N$ *is the quotient map, and*

(e) $\Lambda x = 0$ *for every* $x \in N$*.*

Prove that there is a unique $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ \pi$*, that is,* $\Lambda x = f\big(\pi(x)\big)$ *for all* $x \in X$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

PROOF. Bear in mind that π continuously maps X onto the topological Hausdorff space X/N, since N is closed; see [1.41] of [4]. Moreover, the equation in f_Λ $\Lambda = f_\Lambda \circ \pi$ has necessarily a unique solution f, whose graph is

$$\big\{(\pi x, \Lambda x) : x \in X\big\} \subseteq X/N \times Y. \tag{1.85}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ justifies

$$\pi(x) = \pi(x') \Rightarrow x - x' \in N \Rightarrow \Lambda x - \Lambda x' = 0 \tag{1.86}$$

It straightforwardly derives from (1.85) that f inherits linearity from π and Λ.

Remark. The strengthening $\Lambda x = 0$ **iff** $x \in N$ of (e) is *the first isomorphism theorem* in the topological space context. In this case,

$$f(\pi x) = \Lambda x = 0 \Rightarrow x \in N, \tag{1.87}$$

which implies that f is one-to-one.

Now assume f to be continuous. $\Lambda = f \circ \pi$ is continuous as well, by (a) of [1.41] of [4]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(U) = f \circ \pi(U) \subseteq V. \tag{1.88}$$

Because π is open, by (a) of [1.41] of [4], $\pi(U)$ is a neighborhood of $N = 0_{X/N}$. Therefore, f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by (a) of [1.41] of [4]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$W = \pi(V) \tag{1.89}$$

for some neighborhood V of 0_X. So,

$$f(W) = f \circ \pi(V) = \Lambda(V). \tag{1.90}$$

As a consequence, if Λ is open, then $f(W)$ is a neighborhood of 0_Y. The proof is now complete. □

10 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \to Y$ *is linear, and* $\Lambda(X) = Y$.

(a) Prove that Λ *is an open mapping.*

(b) Assume, in addition, that the null space of Λ *is closed, and prove that* Λ *is continuous.*

PROOF. The case $Y = \{0\}$ is trivial. Let Y have dimension $n > 0$ and basis $B = \{e, e', ...\}$. Let $W \subseteq X$ be a neighborhood of the origin. Since addition is continuous in X, there exists a balanced open V such that

$$\sum_e V \subseteq W. \tag{1.91}$$

Note that $\Lambda(V)$ is balanced as well. Moreover, by the surjectivity of Λ, there exists a vector $x_e \in X$ such that $\Lambda x_e = e$. Therefore, given the coordinate representation $\sum_e y_e e$ of $y \in Y$, we have

$$y = \sum_e y_e e = \sum_e y_e \Lambda x_e. \tag{1.92}$$

Note that the finite set $\{x_e : e \in B\}$ is bounded. Hence

$$\{x_e : e \in B\} \subseteq sV \tag{1.93}$$

for some $s > 0$. Equation (1.92) then yields

$$y \in \sum_e s y_e \Lambda(V). \tag{1.94}$$

Using (1.91) and the balancedness of $\Lambda(V)$, we conclude that $|y_e| < 1/s$ implies

$$y \in \sum_e \Lambda(V) \subseteq \Lambda(W). \tag{1.95}$$

This proves (a) for $Y = \mathbf{C}^n$ equipped with $\|\cdot\|_\infty$ and the standard basis. The general case now follows from [v].

To prove (b), assume that the null space $N = \{x : \Lambda x = 0\}$ is closed and let f, π be as in Exercise 1.9. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. By [v], f is also a homeomorphism. We have thus established that f is continuous; so is $\Lambda = f \circ \pi$. □

12 Exercise 12. Topology stays, completeness leaves

Suppose $d_1(x,y) = |x-y|, d_2(x,y) = |\phi(x) - \phi(y)|$, *where* $\phi(x) = x/(1+|x|)$. *Prove that* d_1 *and* d_2 *are metrics on* $\mathbf{R}$ *which induce the same topology, although* d_1 *is complete and* d_2 *is not.*

PROOF. First, each d_i $(i = 1, 2)$ induces a topology τ_i whose open balls are all

$$B_i(a,r) \triangleq \{x \in \mathbf{R} : d_i(a,x) < r\} \qquad (a \in \mathbf{R}, r > 0). \tag{1.96}$$

Next, remark that the monotonically increasing mapping $\phi : \mathbf{R} \to]\text{-}1, 1[$ is odd and that

$$\phi(x) \xrightarrow[x\to\infty]{} 1. \tag{1.97}$$

ϕ is therefore a τ_1-homeomorphism of $\mathbf{R}$ onto $]\text{-}1, 1[$. A first consequence is that, at fixed $a \in \mathbf{R}$, given any positive scalar ε, the τ_1-continuity of ϕ supplies an open ball $B_1(a,\eta)$ on which $|\phi(a) - \phi| < \varepsilon$. In terms of balls B_i, this reads as follows,

$$B_1(a,\eta) \subseteq B_2(a,\varepsilon). \tag{1.98}$$

The second consequence is that the τ_1-continuity of ϕ^{-1} yields similar inclusions

$$B_2(a,\varepsilon') \subseteq B_1(a,\eta') \tag{1.99}$$

provided $\eta' > 0$. At arbitrary ε, the special case $\eta' = \eta$ is the concatenation

$$B_2(a,\varepsilon') \subseteq B_1(a,\eta) \subseteq B_2(a,\varepsilon); \tag{1.100}$$

which proves that $\tau_1 = \tau_2$. Finally, all inequalities $n < i < j$ over $\mathbf{N}$ together yield

$$d_2(i,j) = |\phi(i) - \phi(j)| \xrightarrow[n\to\infty]{} 0. \tag{1.101}$$

The sequence $n = 0, 1, 2, ...$ is therefore τ_2-Cauchy. We will nevertheless establish that it τ_2-diverges. To do so, we start by assuming the τ_2-convergence to some λ: The triangle inequality immediately dismisses that assumption, as follows,

$$d_2(0,\lambda) \geq d_2(0,n) - d_2(\lambda,n) = \phi(n) - d_2(\lambda,n) \xrightarrow[n\to\infty]{} 1. \tag{1.102}$$

We then conclude that d_2 fails to be complete. □

14 Exercise 14. $\mathcal{D}_K$ equipped with other seminorms

Put $K = [0,1]$ *and define* $\mathcal{D}_K$ *as in Section 1.46. Show that the following three families of seminorms (where* $n = 1, 2, 3, ...$*) define the same topology on* $\mathcal{D}_K$. *If* $D = d/dx$*:*

(a) $\|D^n f\|_\infty = \sup\{|D^n f(x)| : -\infty < x < \infty\}$.

(b) $\|D^n f\|_1 = \int_0^1 |D^n f(x)|\, dx$.

(c) $\|D^n f\|_2 = \left\{\int_0^1 |D^n f(x)|^2\, dx\right\}^{1/2}$.

PROOF. The vector space $C_K = C(\mathbf{R}) \cap \{u : \mathrm{supp}(u) \subseteq K\}$ contains $\mathcal{D}_K$. It is equipped with the inner product

$$\langle u|v\rangle \triangleq \int_0^1 u\bar{v} \qquad \left((u,v) \in {C_K}^2\right). \tag{1.103}$$

In particular, $\langle u|u\rangle = \|u\|_2^2$. The Cauchy-Schwarz inequality with $u = 1$ and $v = |D^n f|$ is then

$$\int_0^1 |D^n f| \leq \|1\|_2 \cdot \|D^n f\|_2; \tag{1.104}$$

see Theorem [12.2] of [4]. Since K has length 1, we obtain

$$\|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty. \tag{1.105}$$

Moreover, the fact that $D^n f(0) = 0$ implies

$$|D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \le \int_0^x |D^{n+1} f| \le \|D^{n+1} f\|_1. \tag{1.106}$$

Taking the supremum of $|D^n f(x)|$ over all x strengthens (1.105), as follows:

$$\|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty \le \|D^{n+1} f\|_1 \le \cdots. \tag{1.107}$$

Finally, set

$$V_n^{(i)} \triangleq \left\{ f : f \in \mathcal{D}_K, \|D^n f\|_i < 2^{-n} \right\} \tag{1.108}$$

for each seminorm i, so that (1.107) is reflected in the decreasing sequence

$$V_n^{(1)} \supseteq V_n^{(2)} \supseteq V_n^{(\infty)} \supseteq V_{n+1}^{(1)} \supseteq \cdots. \tag{1.109}$$

Inclusions $V_n^{(i)} \supseteq V_{n+1}^{(i)}$ turn $\left\{V_n^{(i)}\right\}$ into a local base of a topology τ_i; see [1.37] of [4]. The chain (1.109) forces

$$\tau_1 \subseteq \tau_2 \subseteq \tau_\infty \subseteq \tau_1, \tag{1.110}$$

which completes the argument. □

16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

Lemma 1 Let X be a topological space with a countable local base $\{V_n : n = 1, 2, 3, ...\}$. If $\tilde{V}_n = V_1 \cap \cdots \cap V_n$, then every subsequence $\{\tilde{V}_{\rho(n)}\}$ is a decreasing (equivalently, $\tilde{V}_{\rho(n)} \supseteq \tilde{V}_{\rho(n+1)}$) local base of X.

PROOF. The proof consists in combining trivial consequences of the local base definition with a well-known result (for instance, see [2.6] of [3]) about intersection of nonempty compact sets.

The decreasing property is trivial. Now remark that $V_n \supseteq \tilde{V}_n$: This shows that $\{\tilde{V}_n\}$ is a local base of X. Then so is $\{\tilde{V}_{\rho(n)}\}$, since $\tilde{V}_n \supseteq \tilde{V}_{\rho(n)}$. □

The following special case $V_n = \tilde{V}_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde{V}_n$) Under the same notations of Lemma 1, if $\{V_n\}$ is a decreasing local base, then so is $\{V_{\rho(n)}\}$.

Corollary 2 If $\{Q_n\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{Q_{\rho(n)}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $C(\Omega)$'s (respectively $C^\infty(\Omega)$'s) topology of the seminorms p_n, as defined in section 1.44 (respectively 1.46), then the seminorms $p_{\rho(n)}$ define the same topology τ_Q.

PROOF. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{p_n < 1/n\}$, then $\{V_n\}$ is a decreasing local base of X. Moreover,

$$Q_{\rho(n)} \subseteq Q^\circ_{\rho(n)+1} \subseteq Q_{\rho(n)+1} \subseteq Q_{\rho(n+1)}. \tag{1.111}$$

Thus,

$$Q_{\rho(n)} \subseteq Q^\circ_{\rho(n+1)}. \tag{1.112}$$

In other words, $Q_{\rho(n)}$ satisfies the conditions specified in section 1.44. $\{p_{\rho(n)}\}$ then defines a topology τ_{Q_ρ} for which $\{V_{\rho(n)}\}$ is a local base. So, $\tau_{Q_\rho} \subseteq \tau_Q$. Conversely, the above corollary asserts that $\{V_{\rho(n)}\}$ is a local base of τ_Q, which yields $\tau_Q \subseteq \tau_{Q_\rho}$. □

Lemma 2 If a sequence of compact sets $\{Q_n\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in almost all Q_n°, *i.e.*, there exists m such that

$$K \subseteq Q_m^\circ \subseteq Q_{m+1}^\circ \subseteq Q_{m+2}^\circ \subseteq \cdots. \tag{1.113}$$

PROOF. The following definition

$$C_n \triangleq K \setminus Q_n^\circ \tag{1.114}$$

yields a decreasing sequence of compact[1] sets $\{C_n\}$. Suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q_n°. On the other hand, the conditions specified in [1.44] force the Q_n°'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, , *i.e.*, $K \subseteq Q_m^\circ$, for some m. Finally,

$$K \subseteq Q_m^\circ \subseteq Q_m \subseteq Q_{m+1}^\circ \subseteq Q_{m+1} \subseteq Q_{m+2}^\circ \subseteq \cdots. \tag{1.115}$$

□

We are now in a fair position to establish the following:

Theorem The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.

PROOF. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\kappa}$ for every subsequence $\{L_{\kappa(n)}\}$. Now apply the above Lemma 2 with K_i $(i = 1, 2, 3, ...)$ and so conclude that $K_i \subseteq L_{m_i}^\circ \subseteq L_{m_i+1}^\circ \subseteq \cdots$ for some m_i. In particular, the special case $\kappa_i = m_i + i$ is

$$K_i \subseteq L_{\kappa_i}^\circ. \tag{1.116}$$

We now reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, ...\}$ such that

$$L_j \subseteq K_{\lambda_j}^\circ \tag{1.117}$$

Combine (1.116) with (1.117) and so obtain

$$K_1 \subseteq L_{\kappa_1}^\circ \subseteq L_{\kappa_1} \subseteq K_{\lambda_{\kappa_1}}^\circ \subseteq K_{\lambda_{\kappa_1}} \subseteq L_{\kappa_{\lambda_{\kappa_1}}}^\circ \subseteq \cdots, \tag{1.118}$$

which means that the sequence $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L. \tag{1.119}$$

So ends the proof □

17 Exercise 17. Differentiation in a non-normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathcal{D}_K$ *into* $\mathcal{D}_K$*, for every multi-index* α.

[1] See [(b) of 2.5] of [3].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nonempty compact sets is nonempty. This is a corollary of [2.6] of [3].

PROOF. In both cases, D^α is a linear mapping. It is then sufficient to establish continuity at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an arbitrary neighborhood of the origin. It exists N such that U contains

$$V_N = \left\{\phi \in C^\infty(\Omega) : \max\left\{|D^\beta \phi(x)| : |\beta| \leq N, x \in K_N\right\} < 1/N\right\}. \tag{1.120}$$

Now pick g in $V_{N+|\alpha|}$so that

$$\max\left\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\right\} < \frac{1}{N+|\alpha|}. \tag{1.121}$$

(the fact that $K_N \subseteq K_{N+|\alpha|}$ was tacitly used). The special case $\gamma = \beta + \alpha$ yields

$$\max\left\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\right\} < \frac{1}{N}. \tag{1.122}$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \qquad \text{, } i.e., \qquad D^\alpha\left(V_{N+|\alpha|}\right) \subseteq V_N, \tag{1.123}$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$.

To prove the continuity of the restriction $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \to \mathcal{D}_K$, we first remark that the collection of the $V_N \cap \mathcal{D}_K$ is a local base of the subspace topology of $\mathcal{D}_K$. $V_{N+|\alpha|} \cap \mathcal{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$D^\alpha|_{\mathcal{D}_K}\left(V_{N+|\alpha|} \cap \mathcal{D}_K\right) = D^\alpha\left(V_{N+|\alpha|} \cap \mathcal{D}_K\right) \tag{1.124}$$

$$\subseteq D^\alpha\left(V_{N+|\alpha|}\right) \cap D^\alpha(\mathcal{D}_K) \tag{1.125}$$

$$\subseteq V_N \cap \mathcal{D}_K \qquad \text{(see (1.123))} \tag{1.126}$$

So ends the proof. □

Chapter 2

Completeness

3 Exercise 3. Equicontinuity of Radon measures without vague convergence

Put $K = [-1,1]$; define $\mathcal{D}_K$ as in Section 1.46 (with $\boldsymbol{R}$ in place of $\boldsymbol{R}^n$). Suppose $\{f_n\}$ is a sequence of Lebesgue integrable functions such that $\Lambda(\phi) = \lim_{n\to\infty} \int_{-1}^{1} f_n(t)\phi(t)\,dt$ exists for every $\phi \in \mathcal{D}_K$. Show that Λ is a continuous linear functional on $\mathcal{D}_K$. Show that there is a positive integer p and a number $M < \infty$ such that

$$\left| \int_{-1}^{1} f_n(t)\,\phi(t)\,dt \right| \le M \left\| D^p \phi \right\|_\infty \tag{2.1}$$

for all n. For example, if $f_n(t) = n^3 t$ on $[-1/n, 1/n]$ and 0 elsewhere, show that this can be done with $p = 1$. Construct an example where it can be done with $p = 2$ but not with $p = 1$.

PROOF.

> *Throughout, $\|\cdot\|_\infty$ is denoted by $\|\cdot\|$, without subscript, to set it apart from the derivative norms $\|\cdot\|_N$, which topologize $\mathcal{D}_K$. Also, $\mathbf{R}$ is equipped with its usual Borel measure β, that is, its Lebesgue measure restricted to Borel sets. We keep the usual dt notation when we estimate integrals of smooth functions.*

The integrals $\int_K \phi \cdot f_n \, d\beta$ define linear functionals

$$\begin{aligned} \Lambda_n : \mathcal{D}_K &\to \mathbf{C} \\ \phi &\mapsto \int_K \phi \cdot f_n \, d\beta, \end{aligned} \tag{2.2}$$

whose pointwise limit, as n tends to infinity, is by hypothesis a linear functional Λ. Explicitly,

$$\begin{aligned} \Lambda : \mathcal{D}_K &\to \mathbf{C} \\ \phi &\mapsto \lim_n \Lambda_n(\phi). \end{aligned} \tag{2.3}$$

First, we prove that every Λ_n is continuous with respect to the topology τ_K of $\mathcal{D}_K$. Next, we prove that the limit Λ is τ_K-continuous as well and that there exist a positive order p and a finite bound $M = M_p$ such that

$$|\Lambda_n(\phi)| \le M \left\| D^p \phi \right\| \qquad (\phi \in \mathcal{D}_K) \tag{2.4}$$

for all n. We then exhibit instances of Λ_n whose pointwise limits Λ are not norm-bounded. The proof is split into the following paragraphs:

Continuity of Λ: The limit Λ of $\{\Lambda_n\}$ belongs to the topological dual $\mathcal{D}_K^*$.

Existence of the uniform bound $|\Lambda_n(\phi)| \le M \|D^p \phi\|$: The bound follows from the τ_K-equicontinuity of $\{\Lambda_n\}$.

Existence of $\{\Lambda_n\}$ with optimal uniform bound $|\Lambda_n(\phi)| \le M \|D \phi\|$: The limit Λ does not extend to a Radon measure.

Existence of $\{\Lambda_n\}$ with optimal uniform bound $|\Lambda_n(\phi)| \le M \|D^2\phi\|$, from a polynomial density g_n: The smallest admissible p is 2; this answers the exercise.

Existence of $\{\Lambda_n\}$ with optimal uniform bound $|\Lambda_n(\phi)| \le M \|D^2\phi\|$, no density f_n: The smallest admissible p is 2 again.

Continuity of Λ. Under these conditions, $1_K \cdot f_n$ is identified *modulo* $\mathcal{N}_\beta$ with

$$\begin{aligned} \mu_n : C_c(K) &\to \mathbf{C} \\ u &\mapsto \int_K u \cdot f_n \, d\beta, \end{aligned} \tag{2.5}$$

which is a Radon measure in the sense of [5]. In more detail, the congruence modulo $\mathcal{N}_\beta$ means the following: if a measurable function $g_n : \mathbf{R} \to \mathbf{C}$ satisfies

$$\int_K u \cdot g_n \, d\beta = \int_K u \cdot f_n \, d\beta \qquad \big(u \in C_c(K)\big), \tag{2.6}$$

then

$$1_K \cdot (g_n - f_n) \in \mathcal{N}_\beta. \tag{2.7}$$

Every μ_n is continuous since $\|\mu_n\| = \|1_K \cdot f_n\|_{L^1} < \infty$; see [6.19] of [3]. We introduce

$$\mathcal{C}_K^\infty \triangleq C^\infty(\mathbf{R}) \cap \big\{u : \mathrm{supp}(u) \subseteq K\big\}, \tag{2.8}$$

equipped with the supremum norm $\|\cdot\|$. The canonical injection

$$\begin{aligned} R : \mathcal{C}_K^\infty &\hookrightarrow C_c(K) \\ \phi &\mapsto \phi|_K \end{aligned} \tag{2.9}$$

is an isometry of $\mathcal{C}_K^\infty$ onto a subspace of $C_c(K)$. We recall that $\mathcal{D}_K$ is $\mathcal{C}_K^\infty$ topologized by the *derivative norms*

$$\begin{aligned} \|\cdot\|_N : \mathcal{C}_K^\infty &\to [0, \infty) \\ \phi &\mapsto \max_{0 \le k \le N} \|D^k \phi\| \end{aligned} \tag{2.10}$$

when N ranges over $\mathbf{N}_+$; see [1.46, 6.2] of [4]. In other words, $\mathcal{D}_K$ is $\mathcal{C}_K^\infty$ equipped with τ_K, the weakest vector space topology for which all norms $\|\cdot\|_N$ are continuous. Note that the choice of K, of half-length 1, simplifies each $\|\cdot\|_N$. Indeed, for $\phi \in \mathcal{C}_K^\infty$, Lemma [A.6], with $k = 0, \dots, N$ as N and $N - k$ as p, gives

$$\|D^k \phi\| \le \|D^N \phi\| \frac{1}{(N-k)!} \le \|D^N \phi\|. \tag{2.11}$$

Taking the maximum over k outputs

$$\|\phi\|_N = \|D^N \phi\|. \tag{2.12}$$

It also follows from the definition of $\|\cdot\|_N$ that, in particular,

$$\|\phi\| \le \|\phi\|_1 \qquad (\phi \in \mathcal{D}_K). \tag{2.13}$$

Consequently, every $\|\cdot\|$-open ball $B(r) = \big\{\phi : \|\phi\| < r\big\}$ contains $B_1(r) = \big\{\phi : \|\phi\|_1 < r\big\}$, a τ_K-neighborhood of the origin ($r > 0$). The topology τ_K is then stronger than the $\|\cdot\|$-topology of $\mathcal{C}_K^\infty$. Equivalently, the identity mapping

$$\begin{aligned} I : \mathcal{D}_K &\hookrightarrow \mathcal{C}_K^\infty \\ \phi &\mapsto \phi \end{aligned} \tag{2.14}$$

is continuous. However, I is not bicontinuous. To see this, take the functions ϕ_ω of Lemma [A.7]. For every $\omega > 1/r$, the function ϕ_ω/ω satisfies $\|\phi_\omega/\omega\| = 1/\omega < r$ and $\|D(\phi_\omega/\omega)\| \ge 1$. Thus, it lies in $B(r)$ but outside $B_1(1)$. This

shows that $B_1(1)$ *contains no ball* $B(r)$. The topology τ_K is then *strictly* stronger than the $\|\cdot\|$-topology of $\mathcal{C}^\infty_K$. Additionally, we also see from (2.13) that $\|\phi\|_1 < 1$ implies that $\|\phi\| < 1$. So,

$$|\Lambda_n(\phi)| \le \|f_n\|_{L^1} \qquad \big(\|\phi\|_1 < 1\big). \tag{2.15}$$

As a linear functional that is bounded on a τ_K-neighborhood of 0, Λ_n is τ_K-continuous, by Theorem [1.18] of [4]. Furthermore, the transfer T,

$$\begin{aligned} T : \mathcal{D}_K &\hookrightarrow C_c(K) \\ \phi &\mapsto \phi|_K \end{aligned} \tag{2.16}$$

is a continuous injection, as the composite $R \circ I$ of continuous injections. Thus, our *test function space* $\mathcal{D}_K$ is *continuously embedded* in $C_c(K)$ by T. As in [4.10] of [4], we define the *adjoint* of T as the linear mapping

$$\begin{aligned} T^* : C_c(K)^* &\to \mathcal{D}_K^* \\ \mu &\mapsto \mu \circ T. \end{aligned} \tag{2.17}$$

Here $\mathcal{D}_K$ is not normable, but $\mu \circ T$ is continuous for every continuous μ: T^* is well defined. In particular,

$$\Lambda_n = \mu_n \circ T = T^*(\mu_n), \tag{2.18}$$

or, equivalently,

$$\Lambda_n(\phi) = \mu_n(\phi|_K) = T^*(\mu_n)(\phi) \qquad (\phi \in \mathcal{D}_K). \tag{2.19}$$

Once we identify ϕ with $\phi|_K$, (2.19) allows us to see Λ_n as the τ_K-*continuous restriction* of μ_n. Conversely, the Radon measure μ_n can be regarded as an extension of Λ_n to $C_c(K)$, on which only the strictly weaker norm topology is available. Note that taking adjoints reverses arrows: T maps $\mathcal{D}_K$ into $C_c(K)$, while T^* maps $C_c(K)^*$ into $\mathcal{D}_K^*$. This arrow-reversal pattern, known as *contravariance*, plays an important role in physics and computer science too.[1]

The counterexample in (2.70) furnishes $\Lambda : \phi \mapsto \frac{2}{3} D\phi(0)$, an element of $\mathcal{D}_K^*$ that is not $\|\cdot\|$-continuous; see (2.81). Thus, it cannot be extended to any element μ of $C_c(K)^*$: otherwise, $\Lambda = \mu \circ T$ would be $\|\cdot\|$-bounded. Therefore,

$$T^* : C_c(K)^* \to \mathcal{D}_K^* \text{ is not surjective.} \tag{2.20}$$

In summary, $\mathcal{D}_K^*$ contains the restrictions $T^*(\mu)$ of all Radon measures μ on K, but also linear functionals that do not extend to Radon measures.

To complete the discussion, we prove that the pointwise limit Λ of any sequence as in the statement is always τ_K-continuous. Indeed:

(a) $\mathcal{D}_K$, equipped with the topology τ_K, is a Fréchet space; see [1.46] of [4].

(b) Every functional Λ_n is τ_K-continuous (see above).

(c) $\Lambda_n \longrightarrow \Lambda$ pointwise on $\mathcal{D}_K$ (our premise).

Theorem [2.8] of [4] applies with these three statements, hence the τ_K-continuity of Λ.

Existence of the uniform bound $|\Lambda_n(\phi)| \le M\|D^p\phi\|$. Let ϕ range over $\mathcal{D}_K$. The convex balanced subsets

$$V_N \triangleq \big\{\phi : \|\phi\|_N < 1/N\big\} \qquad (N = 1, 2, 3, \dots) \tag{2.21}$$

form a local base of τ_K, as in [1.46, 6.2] of [4]. For arbitrary N, the definition of V_N gives

$$\frac{1}{N} \cdot \frac{\phi}{\alpha} \in V_N \tag{2.22}$$

[1] In computer science, a function that accepts every argument of a type can replace a function declared for a *subtype* of it. In our case, the injection $T : \mathcal{D}_K \hookrightarrow C_c(K)$ turns $\mathcal{D}_K$ into a subtype of $C_c(K)$, and calling $\mu_n\big(T(\phi)\big)$ returns $\Lambda_n(\phi)$.

for every $\alpha > \|\phi\|_N$. Hence,

$$\phi \in N\alpha V_N. \tag{2.23}$$

Because $\Lambda_n(\phi) \longrightarrow \Lambda(\phi)$ for each ϕ, Theorem [2.6] of [4] implies that $\{\Lambda_n\}$ is equicontinuous. Since $V_1, V_2, \ldots$ is a local base of τ_K, there exists an index p such that, for all n,

$$|\Lambda_n(\phi)| < 1 \qquad \big(\phi \in V_p\big). \tag{2.24}$$

Combining (2.24) with (2.23) yields

$$|\Lambda_n(\phi)| < p \cdot \alpha. \tag{2.25}$$

Since this inequality holds for every $\alpha > \|\phi\|_p$, we have

$$|\Lambda_n(\phi)| \leq p \cdot \|\phi\|_p. \tag{2.26}$$

Combining (2.26) with (2.12) for $N = p$, we obtain

$$|\Lambda_n(\phi)| \leq p \left\| D^p \phi \right\|. \tag{2.27}$$

We have thus proved that (2.4) holds with the uniform bound $M_p = p$, for some $p > 0$. To extend the discussion, we define the smallest admissible order m as the smallest $p \geq 0$ that satisfies (2.4) with some finite M. By definition, m depends only on $\{\Lambda_n\}$. Every $p \geq m$ is admissible as well, since

$$\|\phi\| \leq \left\| D\phi \right\| \leq \left\| D^2 \phi \right\| \leq \cdots, \tag{2.28}$$

by Lemma [A.6] (as in (2.11)). Taking the contrapositive, we see that

$$\text{if, at order p, no finite M satisfies (2.4), then } m > p. \tag{2.29}$$

For instance, if (2.4) fails for $p = 1$, then $m > 1$. In the following, this criterion lets us quickly establish that the order 2 we find for (2.4) is actually the smallest admissible order.

Existence of $\{\Lambda_n\}$ with optimal uniform bound $|\Lambda_n(\phi)| \leq M\|D\phi\|$. Let K_n be the segment $[-1/n, 1/n]$; put $K_n^- = K_n \cap (-\infty, 0]$ and $K_n^+ = K_n \cap [0, \infty)$. We address the example of f_n given in the exercise statement, that is,

$$\begin{aligned} f_n : \mathbf{R} &\to \mathbf{R} \\ t &\mapsto n^3 t\Big[t \in K_n\Big]. \end{aligned} \tag{2.30}$$

The measure μ_n is now the difference of two positive Radon measures μ_n^+ and μ_n^-, as follows:

$$\mu_n(u) = \underbrace{n^3 \int_{K_n^+} u \cdot \mathrm{id}_{\mathbf{R}}\, d\beta}_{\mu_n^+(u)} - \underbrace{n^3 \int_{K_n^-} u \cdot [-\mathrm{id}_{\mathbf{R}}]\, d\beta}_{\mu_n^-(u)} \qquad \big(u \in C_c(K)\big). \tag{2.31}$$

The linear functional μ_n is then a signed Radon measure whose compact support K_n shrinks to $\{0\}$, as n goes to infinity. Furthermore, the sequence $\{\mu_n\}$ is not uniformly bounded, since

$$\|\mu_n\| = \|f_n\|_{L^1} = n, \tag{2.32}$$

$$\|\mu_n^+\| = \|\mu_n^-\| = \frac{n}{2}. \tag{2.33}$$

These equalities follow from [6.19] of [3]. We establish the counterpart for Λ_n, that is,

$$\|\Lambda_n\| = \|f_n\|_{L^1} = n, \tag{2.34}$$

$$\|\Lambda_n^+\| = \|\Lambda_n^-\| = \frac{n}{2}, \tag{2.35}$$

where Λ_n^+ (respectively, Λ_n^-) is the restriction of μ_n^+ (respectively, μ_n^-) to $\mathcal{D}_K$. This is stronger than (2.32): the smaller space $\mathcal{D}_K$ already gives the norm n, since, by definition of the operator norm, $\|\Lambda_n\|$ is the supremum of $|\Lambda_n(\phi)|$ over

all $\phi \in \mathcal{D}_K$ with $\|\phi\| \leq 1$. As a consequence, $\{\Lambda_n\}$, which is seen as the restriction of $\{\mu_n\}$ to $\mathcal{D}_K$, is not uniformly bounded either. The core idea is that the following *logistic function*

$$\begin{aligned} \sigma_\lambda : \mathbf{R} &\to \,]\,0,1\,[\\ t &\mapsto \frac{1}{1+\exp(-\lambda t)}, \end{aligned} \tag{2.36}$$

as $\lambda \longrightarrow \infty$, serves as a standard smooth approximation of

$$\begin{aligned} H : \mathbf{R} &\to \{0,1\} \\ t &\mapsto [\,t \geq 0\,], \end{aligned} \tag{2.37}$$

the Heaviside step function; see [A.3]. More specifically, we asymptotically approximate $\|\mu_n^+\|$, the contribution of H to μ_n, by integrals $\Lambda_n^+(\zeta_\lambda)$ of smooth functions ζ_λ in $\mathcal{D}_K$ (defined below), as follows:

$$\Lambda_n^+(\zeta_\lambda) \xrightarrow[\lambda\to\infty]{} \underbrace{\int_{K_n} H f_n \, d\beta}_{\|\mu_n^+\|}. \tag{2.38}$$

From now on, let $\lambda \geq 2$. To take both μ_n^+ and its "counterweight" μ_n^- into account, we choose the affine combination

$$2\sigma_\lambda - 1 = \tanh \circ \theta_\lambda \qquad \Big(\theta_\lambda(t) \triangleq \lambda t/2\Big), \tag{2.39}$$

as a smooth approximation of $\operatorname{sgn} = H - H \circ (\text{-id}_{\mathbf{R}})$. The construction described in [1.46] of [4] provides a function ϕ, which we now call ψ_λ, such that

(a) $\psi_\lambda \in \mathcal{D}_K$,

(b) ψ_λ is identically 1 on $[\,1/\lambda - 1, 1 - 1/\lambda\,]$,

(c) $0 \leq \psi_\lambda \leq 1$.

This gives the test functions

$$\begin{aligned} \zeta_\lambda : \mathbf{R} &\to \,]\,\text{-}1,1\,[\\ t &\mapsto \psi_\lambda(t) \cdot \tanh\Big(\theta_\lambda(t)\Big). \end{aligned} \tag{2.40}$$

With this choice of ζ_λ, we prove the fundamental convergence

$$0 < n - \Lambda_n(\zeta_\lambda) \xrightarrow[\lambda\to\infty]{} 0. \tag{2.41}$$

Since $\|\zeta_\lambda\| \leq 1$, this estimate gives

$$n \leq \|\Lambda_n\| \leq \|f_n\|_{L^1} = n, \tag{2.42}$$

which is (2.34). Next, Lebesgue's dominated convergence theorem justifies (2.38), since

$$H \cdot f_n \cdot \zeta_\lambda \xrightarrow[\lambda\to\infty]{\text{a.e.}} H \cdot f_n \tag{2.43}$$

(this convergence is pointwise everywhere, except at $t = 1$ when $n = 1$). Hence $\|\Lambda_n^+\| = \|\mu_n^+\|$, as follows:

$$\|\mu_n^+\| \overset{(2.38)}{\leq} \|\Lambda_n^+\| \leq \|\mu_n^+\|. \tag{2.44}$$

We already know that $\|\mu_n^+\| = n/2$ from (2.33). This is also proved by computing the integral in (2.38). In detail,

$$\|\mu_n^+\| = n^3 \int_0^{\frac{1}{n}} t \, dt = \frac{n}{2}. \tag{2.45}$$

A symmetry argument for Λ_n^- gives

$$\|\Lambda_n^+\| = \|\Lambda_n^-\| = \frac{n}{2}, \tag{2.46}$$

which is (2.35). We now prove (2.41).

Because ψ_λ and $\mathrm{id}_{\mathbf{R}} \cdot \tanh \circ \theta_\lambda$ are even, we have, for all n,

$$n - \Lambda_n(\zeta_\lambda) \overset{(2.45)}{=} 2n^3 \int_0^{\frac{1}{n}} t\,dt - 2n^3 \int_0^{\frac{1}{n}} t\,\psi_\lambda(t)\,\tanh\Big(\theta_\lambda(t)\Big)\,dt \tag{2.47}$$

$$= 2n^3 \int_0^{\frac{1}{n}} t\,\Big[1 - \psi_\lambda(t)\,\tanh\Big(\theta_\lambda(t)\Big)\Big]\,dt. \tag{2.48}$$

This makes $n - \Lambda_n(\zeta_\lambda)$ positive. Additionally, for all $n > 1$, the integral reduces to

$$n - \Lambda_n(\zeta_\lambda) = 2n^3 \int_0^{\frac{1}{n}} t\,\Big[1 - \tanh\Big(\theta_\lambda(t)\Big)\Big]\,dt, \tag{2.49}$$

since $\psi_\lambda = 1$ on $[0, 1/n]$ for this n. Moreover, for all $t \in \mathbf{R}$,

$$0 < 1 - \tanh\Big(\theta_\lambda(t)\Big) = 2 - 2\sigma_\lambda(t) < 2\,e^{-\lambda t}. \tag{2.50}$$

We then obtain, when $n > 1$,

$$n - \Lambda_n(\zeta_\lambda) = 2n^3 \int_0^{\frac{1}{n}} t\,\Big[1 - \tanh\Big(\theta_\lambda(t)\Big)\Big]\,dt \tag{2.51}$$

$$< 2n^3 \int_0^{\infty} t\,\Big[1 - \tanh\Big(\theta_\lambda(t)\Big)\Big]\,dt \tag{2.52}$$

$$< 4n^3 \underbrace{\int_0^{\infty} t\,e^{-\lambda t}\,dt}_{1/\lambda^2}. \tag{2.53}$$

If $n = 1$, we split $[0, 1]$ into $[0, 1 - 1/\lambda]$, on which ψ_λ equals 1, and $]1 - 1/\lambda, 1]$. The estimates are similar:

$$1 - \Lambda_1(\zeta_\lambda) = 2 \int_0^{1-\frac{1}{\lambda}} t\,\Big[1 - \tanh\Big(\theta_\lambda(t)\Big)\Big]\,dt + 2 \int_{1-\frac{1}{\lambda}}^{1} t\,\Big[1 - \psi_\lambda(t)\,\tanh\Big(\theta_\lambda(t)\Big)\Big]\,dt \tag{2.54}$$

$$< 2 \int_0^{\infty} t\,\Big[1 - \tanh\Big(\theta_\lambda(t)\Big)\Big]\,dt + 2 \int_{1-\frac{1}{\lambda}}^{1} t\,dt \tag{2.55}$$

$$< 4 \int_0^{\infty} t\,e^{-\lambda t}\,dt + \frac{2}{\lambda}. \tag{2.56}$$

Both cases give a two-sided bound for $n - \Lambda_n(\zeta_\lambda)$, as follows:

$$0 < n - \Lambda_n(\zeta_\lambda) < \frac{4n^3}{\lambda^2} + \frac{2}{\lambda}\big[n = 1\big] \xrightarrow[\lambda\to\infty]{} 0, \tag{2.57}$$

which completes the proof of (2.41). The key observation is that this sequence $\{\mu_n\}$ cannot converge pointwise, or "vaguely", in the language of Radon measure theory: by Theorem [2.6] of [4], this would force $\sup_n \|\mu_n\|$ to be finite, contradicting (2.32).

The sequence$\{\Lambda_n\}$, however, converge pointwise on $\mathcal{D}_K$, and they are uniformly bounded with respect to τ_K. We now prove these two properties. We first consider only *real-valued* ϕ in ù $\mathcal{D}_K$. The mean value theorem implies that

$$\phi(1/n) - \phi(-1/n) = \frac{2}{n}\,D\,\phi(t_n) \tag{2.58}$$

for some $t_n = t_n(\phi) \in \,]\text{-}1/n, 1/n\,[$; see, for instance, Theorem [5.10] of [2]. Moreover, integration by parts yields

$$\Lambda_n(\phi) = \frac{n^3}{2} t^2 \phi(t) \Big|_{\text{-}\frac{1}{n}}^{\frac{1}{n}} - \frac{n^3}{2} \int_{\text{-}\frac{1}{n}}^{\frac{1}{n}} t^2 \, D\,\phi(t)\,dt \tag{2.59}$$

$$= \frac{n}{2}\Big(\phi(1/n) - \phi(\text{-}1/n)\Big) - \frac{n^3}{2} \int_{\text{-}\frac{1}{n}}^{\frac{1}{n}} t^2 \, D\,\phi(t)\,dt \tag{2.60}$$

$$= D\,\phi(t_n) - \frac{n^3}{2} \int_{\text{-}\frac{1}{n}}^{\frac{1}{n}} t^2 \, D\,\phi(t)\,dt. \tag{2.61}$$

For instance, $\phi = \mathrm{id}_{\mathbf{R}} \cdot \psi_2$ has $D\,\phi = 1$ on K_n once $n \geq 2$. In this context, the last equality reduces to

$$\Lambda_n(\phi) = 1 - \frac{n^3}{2} \int_{\text{-}\frac{1}{n}}^{\frac{1}{n}} t^2\,dt = \frac{2}{3}. \tag{2.62}$$

By continuity of $D\,\phi$, this suggests that $\Lambda_n(\phi) \longrightarrow \frac{2}{3} D\,\phi(0)$. We establish this convergence in two steps. First,

$$\Lambda_n(\phi) - \frac{2}{3} D\,\phi(0) = D\,\phi(t_n) - D\,\phi(0) - \frac{n^3}{2} \int_{\text{-}\frac{1}{n}}^{\frac{1}{n}} t^2 \, D\,\phi(t)\,dt + D\,\phi(0) \underbrace{\frac{n^3}{2} \int_{\text{-}\frac{1}{n}}^{\frac{1}{n}} t^2\,dt}_{1/3} \tag{2.63}$$

$$= D\,\phi(t_n) - D\,\phi(0) - \frac{n^3}{2} \int_{\text{-}\frac{1}{n}}^{\frac{1}{n}} t^2 \Big(D\,\phi(t) - D\,\phi(0)\Big)\,dt. \tag{2.64}$$

Next, taking absolute values gives

$$\Big|\Lambda_n(\phi) - \frac{2}{3} D\,\phi(0)\Big| \leq |\,D\,\phi(t_n) - D\,\phi(0)\,| + \frac{1}{3} \max_{\frac{1}{n}K} |\,D\,\phi - D\,\phi(0)\,| \xrightarrow[n\to\infty]{} 0. \tag{2.65}$$

As a result,

$$\Lambda_n(\phi) \xrightarrow[n\to\infty]{} \frac{2}{3} D\,\phi(0) \qquad \Big(\phi \in \mathcal{D}_K : \phi(\mathbf{R}) \subseteq \mathbf{R}\Big). \tag{2.66}$$

In general, every $\phi \in \mathcal{D}_K$ is the combination $\Re(\phi) + i\Im(\phi)$ of the real-valued functions $\Re(\phi)$ and $\Im(\phi)$, both in $\mathcal{D}_K$. The linearity of Λ_n and the continuity of the sum extend this convergence to the whole space $\mathcal{D}_K$. That is,

$$\Lambda_n(\phi) \xrightarrow[n\to\infty]{} \frac{2}{3} D\,\phi(0) \qquad (\phi \in \mathcal{D}_K). \tag{2.67}$$

In distribution theory,

$$\begin{aligned} D\,\delta : \mathcal{D}_K &\to \mathbf{C} \\ \phi &\mapsto \text{-}\,D\,\phi(0) \end{aligned} \tag{2.68}$$

is the *derivative* of the *Dirac measure*

$$\begin{aligned} \delta : \mathcal{D}_K &\to \mathbf{C} \\ \phi &\mapsto \phi(0); \end{aligned} \tag{2.69}$$

see [6.1, 6.9] of [4] and [A.3]. We also show that $D\,\delta$, unlike δ, is not a Radon measure. The previous reasoning proves that

$$\Lambda = \text{-}\frac{2}{3} D\,\delta \in \mathcal{D}_K^*. \tag{2.70}$$

We extend the integration by parts (2.60) to every $\phi \in \mathcal{D}_K$ by applying it to $\Re(\phi)$ and $\Im(\phi)$, then recombining $\phi = \Re(\phi) + i\Im(\phi)$. As $\mathbf{C} \equiv \mathbf{R}^2$, Theorem [5.19] of [2] justifies the following mean value inequality

$$\big|\,\phi(1/n) - \phi(\text{-}1/n)\,\big| \leq \frac{2}{n} \|\,D\,\phi\,\|. \tag{2.71}$$

Therefore, for all $\phi \in \mathcal{D}_K$,

$$|\Lambda_n(\phi)| \le \frac{n}{2}\,|\phi(1/n) - \phi(-1/n)| + \frac{n^3}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} t^2\,|D\,\phi(t)|\,dt \tag{2.72}$$

$$\le \left[1 + \frac{1}{3}\right] \|D\,\phi\|. \tag{2.73}$$

This establishes (2.4) at order $p = 1$, with $M = 4/3$. It remains to prove that no reduction to order $p = 0$ is possible. Suppose, for contradiction, that the following bound

$$M_0 \triangleq \sup\{|\Lambda_n(\phi)| : \|\phi\| \le 1, n \in \mathbf{N}_+\} \tag{2.74}$$

is finite. The contradiction comes from (2.34), which establishes that $n = \|\Lambda_n\| \le M_0$ for all n. Alternatively, a direct counterexample is

$$\phi_n \triangleq \psi_2 \cdot \tilde{\phi}_n, \tag{2.75}$$

where $\tilde{\phi}_n$ is the function ϕ of (A.47) with $\omega = A = 1/\tau = n$, and ψ_2 is the cutoff used in ζ_2; see (2.40). For $n \ge 4$, $\|\phi_n\| = 3/2$ and $\Lambda_n(\phi_n)$ equals

$$\frac{n}{2}\Big(\underbrace{\tilde{\phi}_n(1/n)}_{1} - \underbrace{\tilde{\phi}_n(-1/n)}_{-1}\Big) - \underbrace{\frac{n^3}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} t^2\,D\,\tilde{\phi}_n(t)\,dt}_{\frac{1}{3}n} = \frac{2}{3}n. \tag{2.76}$$

For $n \ge 4$, this contradicts the finiteness of M_0:

$$M_0 \ge \Lambda_n\left(\frac{2}{3}\phi_n\right) = \frac{4n}{9} \xrightarrow[n\to\infty]{} \infty. \tag{2.77}$$

Similarly, we prove that Λ is not bounded with respect to $\|\cdot\|$. To this end, suppose, for contradiction, that the operator norm of Λ, that is,

$$\|\Lambda\| \triangleq \sup\{|\Lambda(\phi)| : \|\phi\| \le 1\}, \tag{2.78}$$

is finite. Pick ϕ_ω from the proof of Lemma [A.7], so that

$$\left\|\frac{1}{\sqrt{\omega}}\phi_\omega\right\| \xrightarrow[\omega\to\infty]{} 0. \tag{2.79}$$

On the other hand, since $\phi_\omega \in \mathcal{D}_K$ and $D\,\phi_\omega(0) = \omega$, we verify that

$$\Lambda\left(\frac{1}{\sqrt{\omega}}\phi_\omega\right) = \frac{2}{3}\sqrt{\omega} \xrightarrow[\omega\to\infty]{} \infty, \tag{2.80}$$

which contradicts the assumed finiteness of $\|\Lambda\|$. Hence

$$\|\Lambda\| = \infty. \tag{2.81}$$

This prevents Λ from extending to a Radon measure in the sense of [5] over $C_c(K)$. We have thus exhibited a sequence of Radon measures μ_n that

(a) does not converge vaguely to any Radon measure, but

(b) converges pointwise on $\mathcal{D}_K$ to a linear functional Λ that is τ_K-continuous; see (2.70).

Two examples follow, with $p = 2$ as the smallest admissible order. The first meets the requirement of the exercise: it is built from integrable functions g_n. The second is an order-2 instance as well, but has no density.

Existence of $\{\Lambda_n\}$ with optimal uniform bound $|\Lambda_n(\phi)| \le M\|D^2\,\phi\|$, from a polynomial density g_n. Legendre polynomial of degree 2 (proof is being reviewed).

Existence of $\{\Lambda_n\}$ with optimal uniform bound $|\Lambda_n(\phi)| \le M \|D^2\phi\|$, no density f_n. We define the *derivative*

$$D\Lambda_n : \mathcal{D}_K \to \mathbf{C} \qquad \phi \mapsto -\Lambda_n(D\phi); \tag{2.82}$$

see [6.9] of [4]. Here $\Lambda_n(D\phi)$ is defined, since $D\phi \in \mathcal{D}_K$. We have proved that every Λ_n is continuous. So is the differentiation operator $D \in \mathcal{L}(\mathcal{D}_K)$; see Exercise [1.17]. Therefore, $D\Lambda_n$ is continuous. Applying (2.67) to $D\phi$ yields

$$D\Lambda_n(\phi) \xrightarrow[n\to\infty]{} -\frac{2}{3} D^2\phi(0), \tag{2.83}$$

and Theorem [2.8] of [4] shows that the limit $\phi \mapsto -\frac{2}{3} D^2\phi(0)$ is τ_K-continuous. Applying (2.73) to $D\phi$ in place of ϕ gives the following version of (2.4):

$$|D\Lambda_n(\phi)| \le \frac{4}{3} \|D^2\phi\|. \tag{2.84}$$

We additionally establish that 2 is the smallest admissible order. Suppose, for contradiction, that

$$M_1 \triangleq \sup\{|\Lambda_n(D\phi)| : \|D\phi\| \le 1, n \in \mathbf{N}_+\} \tag{2.85}$$

is finite. Let

$$\Phi_n : \mathbf{R} \to \mathbf{R} \qquad t \mapsto \int_{-1}^{t} \phi_n(u)\,du; \tag{2.86}$$

see (2.75) for the definition of ϕ_n. In other words, $\Phi_n \in C^\infty(\mathbf{R})$ is the antiderivative of ϕ_n that vanishes at -1. Since ϕ_n has support in $[-1,1]$, we obtain

$$\Phi_n(t) = \begin{cases} 0 & (t < -1), \\ \Phi_n(1) & (t > 1). \end{cases} \tag{2.87}$$

Note that ϕ_n is odd, which forces

$$\Phi_n(1) = \int_{-1}^{1} \phi_n(u)\,du = 0. \tag{2.88}$$

Our Φ_n is then identically zero on $(-\infty,-1[\,\cup\,]1,\infty)$. Hence

$$\operatorname{supp}(\Phi_n) \subseteq [-1,1]. \tag{2.89}$$

We have just proved that $\Phi_n \in \mathcal{D}_K$, which allows us to estimate $D\Lambda_n(\Phi_n)$. Since M_1 is assumed to be finite, we have

$$|D\Lambda_n(\Phi_n)| = |\Lambda_n(D\Phi_n)| \le M_1 \|D\Phi_n\| < \infty. \tag{2.90}$$

Equivalently,

$$|\Lambda_n(\phi_n)| \le M_1 \|\phi_n\| < \infty, \tag{2.91}$$

which is disproved as in (2.77). Therefore, $M_1 = \infty$: no bound of order 1 holds for $\{D\Lambda_n\}$. This sequence meets the simple criterion stated right after (2.29). Therefore, the smallest admissible order for $\{D\Lambda_n\}$ is at least 2. With the bound of order 2 above, we conclude that 2 is the minimal order for $\{D\Lambda_n\}$.

□

6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ *of a function* $f \in L^2(T)$ *(*T *is the unit circle) by*

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all $n \in \mathbf{Z}$ *(the integers). Put*

$$\Lambda_n f = \sum_{k=-n}^{n} \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n\infty} \Lambda_n f \text{ exists}\}$ *is a dense subspace of* $L^2(T)$ *of the first category.*

PROOF. Let $f(\theta)$ stand for $f(e^{i\theta})$so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi,\pi])$, hence the inner product

$$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta. \tag{2.92}$$

We express $\Lambda_n(f)$ as follows:

$$\Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n). \tag{2.93}$$

Moreover, a well-known (and easy to prove) result is

$$(e_n, e_{n'}) = [n = n'], \text{ , } \textit{i.e.}, \{e_n : n \in \mathbf{Z}\} \text{ is an orthonormal subset of } L^2(T). \tag{2.94}$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $L^2 \equiv (L^2)^*$. So,

$$\| \Lambda_n \|^2 \overset{2.93}{=} \| e_{-n} + \cdots + e_n \|^2 \overset{2.94}{=} \| e_{-n} \|^2 + \cdots + \| e_n \|^2 \overset{2.94}{=} 2n + 1. \tag{2.95}$$

Now suppose, to reach a contradiction, that

$$B \triangleq \{f \in L^2(T) : \sup\{| \Lambda_n f | : n = 1, 2, 3, ...\} < \infty\} \tag{2.96}$$

is of the second category. So, the Banach-Steinhaus theorem [2.5] of [4] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$\| \Lambda_n \| \overset{2.95}{=} \sqrt{2n+1} \xrightarrow[n\to\infty]{} \infty. \tag{2.97}$$

This establishes that B is actually of the first category; and so is its subset L=$\{f \in L^2(T) : \lim_{n\longrightarrow\infty} \Lambda_n f \text{ exists}\}$. It remains to prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be span$\{e_k : k \in Z\}$, the collection of the trigonometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$: Combining (2.93) with (2.94) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n. Thus,

$$P \subseteq L \subseteq L^2(T). \tag{2.98}$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.98), that

$$L^2(T) = \overline{P} = \overline{L}. \tag{2.99}$$

So ends the proof □

9 Exercise 9. Boundedness without closedness

Suppose X, Y, Z *are Banach spaces and*

$$B : X \times Y \to Z$$

is bilinear and continuous. Prove that there exists $M < \infty$ *such that*

$$\| B(x,y) \| \leq M \| x \| \| y \| \qquad (x \in X, y \in Y).$$

Is completeness needed here?

PROOF. Completeness is not required. To show this, we only assume that X, Y, and Z are normed spaces. Since B is continuous, there exists $r > 0$ such that

$$\|B(x,y)\| < 1 \tag{2.100}$$

whenever $\|x\| + \|y\| < r$. Now, for general $(x,y) \in X \times Y$ and for all $0 < r - s < \min(1,r)$, we substitute:

$$x_s \triangleq \frac{s}{2} \cdot \frac{x}{\|x\| + r - s} \qquad \text{(for x)}, \tag{2.101}$$

$$y_s \triangleq \frac{s}{2} \cdot \frac{y}{\|y\| + r - s} \qquad \text{(for y)}, \tag{2.102}$$

in (2.100). A straightforward computation shows that, at fixed (x,y),

$$\|B(x,y)\| = \frac{4}{s^2}\big(\|x\| + r - s\big)\big(\|y\| + r - s\big)\underbrace{\|B(x_s,y_s)\|}_{<1} < \frac{4}{s^2}\big(\|x\|\,\|y\| + O(r-s)\big) \xrightarrow[s\to r]{} \frac{4}{r^2}\|x\|\,\|y\|\,; \tag{2.103}$$

which establishes the existence of a bound $M = 4/r^2$. To illustrate this result, we now turn to a concrete example.

Take $X = Y = Z = C_c(\mathbf{R})$ equipped with $\|\cdot\|_\infty$, which is noncomplete; see [5.4.4] of [5]. Nevertheless, the following bilinear mapping

$$\begin{aligned} B : C_c(\mathbf{R})^2 &\to C_c(\mathbf{R}) \\ (f,g) &\mapsto f \cdot g \end{aligned} \tag{2.104}$$

is continuous. In addition, B has continuity bound $M = 1$, since

$$\|f \cdot g\|_\infty \le \|f\|_\infty \cdot \|g\|_\infty. \tag{2.105}$$

To prove the continuity of B at every (f,g), fix $\varepsilon > 0$ and let r satisfy

$$0 < r < \min\left(1, \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty}\right). \tag{2.106}$$

Next, choose $(u,v) \in C_c(\mathbf{R})^2$ such that

$$\underbrace{\|u\|_\infty - \|f\|_\infty \le \|f-u\|_\infty}_{\text{reversed triangle inequality}} + \|g - v\|_\infty < r. \tag{2.107}$$

Hence

$$\|\overbrace{(f-u)g + u(g-v)}^{fg-uv}\|_\infty \le \|f-u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g-v\|_\infty \quad \big(\text{by the triangle inequality and (2.105)}\big) \tag{2.108}$$

$$< r \cdot \|g\|_\infty + \big(r + \|f\|_\infty\big) \cdot r \quad \big(\text{since } \|u\| \le r + \|f\|\text{, see (2.107)}\big) \tag{2.109}$$

$$< \varepsilon \frac{r + \|f\|_\infty + \|g\|_\infty}{1 + \|f\|_\infty + \|g\|_\infty} \quad \big(\text{by (2.106)}\big) \tag{2.110}$$

$$< \varepsilon. \quad \text{(since } r < 1\text{, see (2.106))} \tag{2.111}$$

In conclusion,

$$\|B(f,g) - B(u,v)\|_\infty < \varepsilon. \tag{2.112}$$

Since ε was arbitrary, the continuity of B is established. We have thus constructed a continuous bilinear $B : X^2 \to X$ over a noncomplete normed space X. This concludes the proof. □

10 Exercise 10. Continuity of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

PROOF. Let $B : X_1 \times X_2 \to Z$ be a bilinear mapping that is continuous at $(0,0)$, where X_i $\big(i \in \{1,2\}\big)$ and Z are topological vector spaces. This implies that, for any balanced open W, X_i contains a balanced open U_i such that

$$B(U_1 \times U_2) \subseteq W. \tag{2.113}$$

Let a_i be in X_i. Therefore, $a_i \in r_i U_i$ for some positive r_i. We now choose b_i in $a_i + (1 + r_1 + r_2)^{-1} U_i$. Hence

$$B(b_1,b_2) - B(a_1,a_2) = B(b_1 - a_1, b_2) + B(a_1, b_2) - B(a_1,a_2) \tag{2.114}$$

$$= B(b_1 - a_1, b_2) + B(a_1, b_2 - a_2) \tag{2.115}$$

$$= \underbrace{B(b_1 - a_1, b_2 - a_2)}_{\in \frac{1}{(1+r_1+r_2)^2} W} + \underbrace{B(b_1 - a_1, a_2)}_{\in \frac{r_2}{1+r_1+r_2} W} + \underbrace{B(a_1, b_2 - a_2)}_{\in \frac{r_1}{1+r_1+r_2} W} \tag{2.116}$$

$$\in W + W + W. \tag{2.117}$$

We conclude that B is continuous at every (a_1, a_2), since W was arbitrary. □

12 Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)|\,dt.$$

Put $B(f,g) = \int_0^1 f(t)g(t)\,dt$, and show that B is a bilinear functional on $X \times X$ which is separately continuous but is not continuous.

PROOF. Observe that

$$|B(f,g)| \le \max_{[0,1]} |g| \, \|f\|, \tag{2.118}$$

which shows that the partial mapping $f \mapsto B(f,g)$ is continuous; see [1.18] of [4]. By symmetry, the partial mapping $g \mapsto B(f,g)$ is likewise continuous. This concludes the first part. We now prove, however, that B is not continuous. To this end, suppose, for a contradiction, that B is continuous. Applying [2.9] with $X = Y = \mathbf{R}[T]$ and $Z = \mathbf{R}$ (completeness is not needed) establishes that

$$|B(f,f)| \le M \|f\|^2 \qquad (f \in X) \tag{2.119}$$

for some $M < \infty$. Therefore, evaluating at

$$f_n(T) \triangleq \sqrt{n}\, T^n \qquad (n = 1,2,3,\dots) \tag{2.120}$$

yields the desired contradiction:

$$\frac{1}{3} \le B(f_n,f_n) = \frac{n}{2n+1} \le M \|f_n\|^2 = \frac{Mn}{(n+1)^2} \xrightarrow[n\to\infty]{} 0. \tag{2.121}$$

Alternatively, without relying on [2.9], we observe that $\|f_n\| \to 0$, which implies that (f_n, f_n) tends to $(0,0)$ in the product space $X \times X$. In contrast, $B(f_n,f_n) \to 1/2 \neq B(0,0) = 0$. Thus, B is not sequentially continuous and, consequently, not continuous. This completes the proof. □

15 Exercise 15. Baire's cut

Suppose X is an F-space and Y is a subspace of X whose complement is of the first category. Prove that $Y = X$. Hint: *Y must intersect $x + Y$ for every $x \in X$.*

PROOF. Let $\{E_n : n = 1,2,3,\dots\}$ be a collection whose union is $X \setminus Y$. Additionally, assume that every E_n is nowhere dense, *i.e.*, $V_n = X \setminus \overline{E_n}$ is dense in X. Now pick an arbitrary $x \in X$. Since the translation by x is a self-homeomorphism, $x + V_n$ is open and dense[2].We now apply Baire's theorem twice to establish that

[2]Alternatively, observe that $X = x + X \subseteq \overline{x + V_n}$, as a special case of [1.3 (b)] of [4].

(a) every intersection $W_n = V_n \cap (x + V_n)$ is dense in X,

(b) so is the nonempty intersection $\bigcap_{n=1}^{\infty} W_n$.

Moreover, the intersection $\bigcap_{n=1}^{\infty} V_n$ is disjoint from every E_n. In summary,

$$w \in \bigcap_{n=1}^{\infty} W_n \subseteq \bigcap_{n=1}^{\infty} V_n \subseteq Y \tag{2.122}$$

for some $w = w(x)$. Furthermore, w also lies in every $x + V_n$, by (a). This implies

$$w - x \in \bigcap_{n=1}^{\infty} V_n \subseteq Y. \tag{2.123}$$

Finally, (2.122) and (2.123) together yield

$$x = w - (w - x) \in Y - Y = Y, \tag{2.124}$$

because Y is a subgroup of X. This establishes that

$$X \subseteq Y, \tag{2.125}$$

since x was arbitrary. □

16 Exercise 16. An elementary closed graph theorem

Suppose that X *and* K *are metric spaces, that* K *is compact, and that the graph of* $f : X \to K$ *is a closed subset of* $X \times K$. *Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of* K *cannot be omitted from the hypotheses, even when* X *is compact.*

PROOF. Choose a sequence $\{x_n : n = 1, 2, 3, ...\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some (a, b) of $X \times K$. G is closed; therefore, $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ converges in G. So, $b = f(a)$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous, see [[A6]] of [4]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we present the following counterexample,

$$\begin{aligned} f : [0, \infty) &\to [0, \infty) \\ x &\mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases} \end{aligned} \tag{2.126}$$

Clearly, f has a discontinuity at 0. In contrast, the graph G of f is closed. To see that, first remark that

$$G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}. \tag{2.127}$$

Next, let $\{(x_n, 1/x_n)\}$ be a sequence in $G_+ = \{(x, 1/x) : x > 0\}$ that converges to (a, b). To be more specific: $a = 0$ contradicts the boundedness of $\{(x_n, 1/x_n)\}$: a is necessarily positive and $b = 1/a$, since $x \mapsto 1/x$ is continuous on R_+. This establishes that $(a, b) \in G_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. □

Chapter 3

Convexity

3 Exercise 3. Application of the Hahn–Banach theorem

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subseteq X$ *an internal point of* A *if* $A - x_0$ *is an absorbing set.*

(a) Suppose A *and* B *are disjoint convex sets in* X*, and* A *has an internal point. Prove that there is a nonconstant functional* Λ *such that* $\Lambda(A) \cap \Lambda(B)$ *contains at most one point. (The proof is similar to that of Theorem 3.4)*

(b) Show (with $X = \mathbf{R}^2$*, for example) that it may not possible to have* $\Lambda(A)$ *and* $\Lambda(B)$ *disjoint, under the hypotheses of (a).*

PROOF. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ ($b_0 \in B$). Note that C contains the origin. Let p be the Minkowski functional of C. Since A and B are disjoint, $b_0 - x_0$ is not in C, hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem [3.4] of [4] to establish the existence of a functional $\Lambda : X \to \mathbf{R}$ such that

$$\Lambda \leq p \tag{3.1}$$

and

$$\Lambda(b_0 - x_0) = 1. \tag{3.2}$$

Then

$$\Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \qquad (a \in A, b \in B). \tag{3.3}$$

Hence

$$\Lambda a \leq \Lambda b. \tag{3.4}$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2. There exists (a_i, b_i) in $A \times B$ ($i = 1, 2$) such that

$$\Lambda a_i = \Lambda b_i = y_i. \tag{3.5}$$

Assume without loss of generality that $y_1 < y_2$. Then,

$$2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \qquad . \tag{3.6}$$

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A. This implies

$$\Lambda b_1 \overset{(3.6)}{<} \Lambda a_3 \overset{(3.4)}{\leq} \Lambda b_1 \qquad ; \tag{3.7}$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is $\mathbf{R}^2$. Fetch

$$S_1 \triangleq \{(x,y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\}, \tag{3.8}$$

$$S_2 \triangleq \{(x,y) \in \mathbf{R}^2 : x > 0, y > 0\}, \tag{3.9}$$

$$A \triangleq S_1 \cup S_2, \tag{3.10}$$

$$B \triangleq X \setminus A. \tag{3.11}$$

Pick (x_i, y_i) in S_i. Let t range over the unit interval, and so obtain

$$t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subseteq A. \tag{3.12}$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A. Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.*, A° is nonempty[1]: A contains an internal point.

Let L be a vector line of $\mathbf{R}^2$. In other words, L is the null space of a functional $\Lambda : \mathbf{R}^2 \to \mathbf{R}$ (to see this, take some nonzero u in $L^\perp$ and set $\Lambda x = (x,u)$ for all x in $\mathbf{R}^2$). One easily checks that both A and B cut L. Hence

$$\Lambda(L) = \{0\} \subseteq \Lambda(A) \cap \Lambda(B) \neq \emptyset \qquad . \tag{3.13}$$

So ends the proof. □

11 Exercise 11. Meagerness of the polar

Let X *be an infinite-dimensional Fréchet space. Prove that* X^**, with its weak*-topology, is of the first category in itself.*

This is actually a consequence of the lemma below, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. *f X is an infinite dimensional topological vector space whose dual* X^* *separates points on* X*, then the polar*

$$K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\} \tag{3.14}$$

of any absorbing subset A is a weak-closed set that has empty interior.*

PROOF. Let x range over X. The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous, see [3.14] of [4]. Therefore, $P_x = \left\{\Lambda \in X^* : |\Lambda x| \leq 1\right\}$ is weak*- closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see [3.11] of [4]). Put

$$W_{F,x} \triangleq \bigcap_{x \in F} \left\{\Lambda \in X^* : |\Lambda x| < r_x\right\} \qquad (r_x > 0) \tag{3.15}$$

where F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X^*. Pick one of those W and remark that the following subspace

$$M \triangleq \operatorname{span}(F) \tag{3.16}$$

is finite dimensional. Assume, to reach a contradiction, that $A \subseteq M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M, which is a desired contradiction. We have just established that $A \not\subset M$: Now pick *a* in $A \setminus M$ and so conclude that

$$b \triangleq \frac{a}{t_a} \in A \tag{3.17}$$

[1]For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] of [4] or [2.5] of [3].

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M, as a finite dimensional space, is closed (see [1.21 (b)] of [4] for a proof): By the Hahn-Banach theorem [3.5] of [4], there exists Λ_a in X^* such that

$$\Lambda_a b > 2 \tag{3.18}$$

and

$$\Lambda_a(M) = \{0\}. \tag{3.19}$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

$$|\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1. \tag{3.20}$$

show that $\Lambda + \Lambda_a$ is not in K_A. We have thus proved that

$$\Lambda + W \not\subset K_A. \tag{3.21}$$

Since W and Λ are both arbitrary, this achieves the proof. □

We now give a proof of the original statement.

Corollary. *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

PROOF. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

$$B_n \triangleq \{x \in X : d(0,x) < 1/n\} \qquad (n = 1, 2, 3, \dots) \tag{3.22}$$

form a local base of X. If Λ is in X^*, then

$$|\Lambda| \leq m \text{ on } B_n \tag{3.23}$$

for some $(n, m) \in \{1, 2, 3, \dots\}^2$, see [1.18] of [4]. Hence, X^* is the countable union of all

$$m \cdot K_n \qquad (m, n = 1, 2, 3, \dots), \tag{3.24}$$

where K_n is the polar of B_n. Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see [3.4] of [4]. As a consequence, the above lemma implies

$$\big(\overline{K_n}\big)^\circ = (K_n)^\circ = \emptyset. \tag{3.25}$$

Since the multiplication by m is a homeomorphism (see [1.7] of [4]), this is equivalent to

$$\big(\overline{m \cdot K_n}\big)^\circ = m \cdot (K_n)^\circ = \emptyset. \tag{3.26}$$

So ends the proof. □

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

1 Exercise 1. Basic results

Let ϕ be the embedding of X *into* X^{**} *described in Section 4.5. Let τ be the weak topology of* X*, and let σ be the weak*- topology of* X^{**}*- the one induced by* X^**.*

1. *Prove that ϕ is a homeomorphism of (X, τ) onto a dense subspace of (X^{**}, σ).*
2. *If* B *is the closed unit ball of* X*, prove that $\phi(B)$ is σ-dense in the closed unit ball of* X^{**}*. (Use the Hahn-Banach separation theorem.)*
3. *Use (a), (b), and the Banach-Alaoglu theorem to prove that* X *is reflexive if and only if* B *is weakly compact.*
4. *Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.*
5. *If* X *is reflexive and* Y *is a closed subspace of* X*, prove that* X/Y *is reflexive.*
6. *Prove that* X *is reflexive if and only* X^* *if reflexive.*
   Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\phi(X)$ of X^{**}*.*

PROOF. Let ψ be the isometric embedding of X^* into X^{***}. The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

$$\phi^{-1} : \phi(X) \to X \tag{4.1}$$

$$\phi(x) \mapsto x \tag{4.2}$$

is a homeomorphism (with respect to τ and σ). We first consider

$$V \triangleq \{x^{**} \in X^{**} : |\langle x^{**} | \psi x^* \rangle| < r\} \qquad (x^* \in X^*, r > 0); \tag{4.3}$$

$$U \triangleq \{x \in X : |\langle x | x^* \rangle| < r\} \qquad (x^* \in X^*, r > 0). \tag{4.4}$$

and remark that the so defined V 's (respectively U 's) shape a local subbase $\mathcal{S}_\sigma$ (respectively $\mathcal{S}_\tau$) of σ (respectively τ). We now observe that

$$U = \phi^{-1}(V \cap \phi(X)) = \phi^{-1}(V) \cap X \qquad (V \in \mathcal{S}_\sigma,\ U \in \mathcal{S}_\tau) \qquad , \tag{4.5}$$

since ϕ^{-1} is one-to-one. This remains true whether we enrich each subbase $\mathcal{S}$ with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that ϕ^{-1} and its inverse ϕ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$\overline{\phi(X)}_\sigma \subseteq X^{**} \qquad . \tag{4.6}$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ-closure of $\phi(X)$. By [3.5], there exists y^* in X^* such that

$$\langle \phi x, \psi y^* \rangle = \langle y^*, \phi x \rangle = \langle x, y^* \rangle = 0 \qquad (x \in X) \qquad ; \tag{4.7}$$

$$\langle z^{**}, \psi y^* \rangle = 1 \tag{4.8}$$

(4.7) forces y^* to be a the zero of X^*. The functional ψy^* is then the zero of X^{***}: (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$\phi(B) \subseteq B^{**} \qquad , \tag{4.9}$$

since ϕ is isometric. Hence

$$\overline{\phi(B)}_\sigma \subseteq \overline{(B^{**})}_\sigma = B^{**} \qquad . \tag{4.10}$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\phi(B)}_\sigma$ contains a vector z^{**}. By [3.7], there exists y^* in X^* such that

$$|\psi y^*| \leq 1 \qquad \text{on } \overline{\phi(B)}_\sigma \qquad ; \tag{4.11}$$

$$\langle z^{**}, \psi y^* \rangle > 1 \qquad . \tag{4.12}$$

It follows from (4.11) that

$$|\psi y^*| \leq 1 \text{ on } \phi(B), \text{ ie, } |y^*| \leq 1 \text{ on } B \qquad . \tag{4.13}$$

We have so proved that

$$y^* \in B^* \qquad . \tag{4.14}$$

Since z^{**} lies in B^{**}, it is now clear that

$$|\langle z^{**}, \psi y^* \rangle| \leq 1 \qquad ; \tag{4.15}$$

what it contradicts (4.12), and thus proves (b). We now prove (c).

It follows from (a) that

$$\text{B is weakly compact if and only if } \phi(B) \text{ is weak*-compact.} \tag{4.16}$$

If B is weakly compact, then $\phi(B)$ is weak*-closed. So,

$$\phi(B) = \overline{\phi(B)}_\sigma \overset{(b)}{=} B^{**} \qquad . \tag{4.17}$$

ϕ is therefore onto, *i.e.*, X is reflexive.
Conversely, keep ϕ as onto: one easily checks that $\phi(B) = B^{**}$. The image $\phi(B)$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X, whose closed unit ball is B. Let Y be a norm-closed subspace of X: Y is then weakly closed (*cf.*[3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $B \cap Y$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let $\equiv$ stand for "isometrically isomorphic" and apply twice [4.9] to obtain, first

$$(X/Y)^* \equiv Y^\perp \qquad , \tag{4.18}$$

next,

$$(X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y \qquad . \tag{4.19}$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

Given a reflexive Banach space Z*, the weak*-topology of* Z^* *is its weak one.*

$Z^*_{w^*} = Z^*_w$.
Assume first that X is reflexive. Since B^* is weak* compact, by (c) of [4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^* into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\phi(X)$ is a norm-closed subspace of X^{**}; *cf.*[4.5]. Hence $\phi(X)$ is reflexive, by (d). It now follows from (c) that $B^{**} \cap \phi(X)$ is weakly compact, *i.e.*, weak*-compact (to see this, apply (L) with $Z = X^*$).

By (a), B is therefore weakly compact, *i.e.*, X is reflexive, see (c). So ends the proof. □

13 Exercise 13. Operator compactness in a Hilbert space

(a) Suppose $T \in \mathcal{B}(X, Y), T_n \in \mathcal{B}(X, Y)$ *for* $n = 1, 2, 3, \ldots$, *each* T_n *has finite-dimensional range, and* $\lim \|T - T_n\| = 0$. *Prove that* T *is compact.*

(b) Assume Y *is a Hilbert space, and prove the converse of (a): Every compact* $T \in \mathcal{B}(X, Y)$ *can be approximated in the operator norm by operators with finite-dimensional ranges. Hint: In a Hilbert space there are linear projections of norm 1 onto any closed subspace. (See theorems 5.16, 12.4.)*

PROOF. Since each T_n is compact, (a) follows from (c) of [4.18]. Besides, we take the opportunity to alternatively prove that the compact operators subspace is norm closed.

Reset every T_n as a compact operator. Let $\{x_0^i : i \in \mathbf{N}\}$ be in U the open unit ball of X. Since T_1 is compact, $\{x_0^i\}$ contains a subsequence $\{x_1^i : i \in \mathbf{N}\}$ such that $\{T_1 x_1^i\}$ converges to a point y_1 of Y. The same reasoning can be recursively applied to T_n and $\{x_{n-1}^i\} \subseteq U$ so that $\{T_n x_n^i\}$ tends to some y_n of Y, as $\{x_n^i\}$ is a subsequence of $\{x_{n-1}^i\}$. Then

$$T_n x_p^i \underset{i\to\infty}{\longrightarrow} y_n \qquad (p\, n = 1, 2, 3, \cdots) \qquad . \tag{4.20}$$

Applied with $\{x_n^i : (n, i) \in \mathbf{N}^2\}$, a Cantor's diagonal process therefore provides a subsequence $\{\tilde{x}_j : j \in \mathbf{N}\}$ such that

$$T_j \tilde{x}_k \underset{k\to\infty}{\longrightarrow} y_j \qquad ; \tag{4.21}$$

$$T_j \tilde{x}_j \underset{j\to\infty}{\longrightarrow} y_j \qquad . \tag{4.22}$$

We now easily obtain

$$\|T_j \tilde{x}_j - T_k \tilde{x}_k\| \le \|T_j \tilde{x}_j - y_j\| + \|y_j - T_j \tilde{x}_k\| + \|T_j - T_k\| \underset{k>j\to\infty}{\longrightarrow} 0 \qquad . \tag{4.23}$$

$\{T_j \tilde{x}_j\}$ is then a Cauchy sequence. So is $\{T \tilde{x}_j\}$, since $\|T - T_j\| \to 0$. On the other hand, Y is complete: (a) is then proved and we now establish the counterpart in a Hilbert space.

Fix ε as a positive scalar. Since T is compact, Y contains a finite set C such that

$$T(U) \subseteq \bigcup_{c\in C} B(c, \varepsilon) \qquad . \tag{4.24}$$

As a Hilbert space, Y contains a maximal orthonormal set (or Hilbert basis) M. This implies that span(M) is dense in Y; *cf.*4.18 & [4.22] of [3]. The finiteness of C forces M to enclose a finite set S so that

$$\forall c \in C,\ \exists s(c) \in \mathrm{span}(S) : \|c - s(c)\| < \varepsilon \qquad . \tag{4.25}$$

Let x be in U. It follows from (4.24) that

$$\|Tx - c_x\| < \varepsilon \tag{4.26}$$

for some c_x of C. We now combine (4.25) and (4.26) to obtain

$$\|Tx - s(c_x)\| \le \|Tx - c_x\| + \|c_x - s(c_x)\| < 2\varepsilon \tag{4.27}$$

As a finite-dimensional subspace, span(S) is closed (see footnote 4, Exercise 1.10). We so obtain

$$Y = \text{span}(S) \oplus \text{span}(S)^{\perp} \qquad , \tag{4.28}$$

by [12.4]. There exists a unique projection $\pi = \pi(\varepsilon)$ of Y onto itself (see [5.6] for the definition) such that

$$\pi(Y) = \text{span}(S)\,,\ (I-\pi)(Y) = \text{span}(S)^{\perp} \qquad . \tag{4.29}$$

It is easily checked that π has norm 1. Moreover,

$$\pi s = s \qquad (s \in \text{span}(S)) \qquad . \tag{4.30}$$

Thus,

$$(I-\pi)(Tx) = (I-\pi)(Tx - s(c_x)) \qquad (x \in U) \qquad . \tag{4.31}$$

Then,

$$\|(I-\pi)(Tx)\| \le \|I-\pi\|\,\|Tx - s(c_x)\| < 4\varepsilon \qquad (x \in U) \tag{4.32}$$

(the fact that π has norm 1 is hidden in the right side inequality). We have just so proved that

$$\|T - \pi \circ T\| \in \underset{\varepsilon\sim 0}{O}(\varepsilon) \qquad . \tag{4.33}$$

That is particularly true when $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$. Let so T_n be $\pi(\varepsilon_n)\circ T$ and conclude that these (compact) operators approximate T in the desired fashion, *i.e.*,

$$\|T - T_n\| \underset{n\to\infty}{\longrightarrow} 0 \qquad . \tag{4.34}$$

□

15 Exercise 15. Hilbert–Schmidt operators

Suppose μ *is a finite (or* σ*-finite) positive measure on a measure space* Ω, $\mu \times \mu$ *is the corresponding product measure on* $\Omega \times \Omega$, *and* $K \in L^2(\mu \times \mu)$. *Define*

$$(Tf)(s) = \int_{\Omega} K(s,t) f(t) d\mu(t) \qquad [f \in L^2(\mu)].$$

(a) Prove that $T \in \mathcal{B}(L^2(\mu))$ *and that*

$$\|T\|^2 \le \int_{\Omega}\int_{\Omega} |K(s,t)|^2 d\mu(s) d\mu(t).$$

(b) Suppose a_i, b_i *are members of* $L^2(\mu)$, *for* $1 \le i \le n$, *put* $K_1 = \sum a_i(s) b_i(t)$ *and define* T_1 *in terms of* K_1 *a* T *was defined in terms of* K. *Prove that* $\dim \mathcal{R}(T_1) \le n$.

(c) Deduce that T *is a compact operator in* $L^2(\mu)$. *Hint: Use exercise 13.*

(d) Suppose $\lambda \in \mathbf{C}$, $\lambda \neq 0$. *Prove: Either the equation*

$$Tf - \lambda f = g$$

has a unique solution $f \in L^2(\mu)$ *for every* $g \in L^2(\mu)$ *or there are infinitely many solutions for some* g *and none for others. (This is known as the* Fredholm alternative.*).*

(e) Describe the adjoint of T.

PROOF. Let X (respectively P) be the Banach space $L^2(\mu)$ (respectively $L^2(\mu \times \mu)$). A consequence of the Radon-Nikodym theorem (*cf.*[6.16] of [3]) is that there exists a group isomorphism $\rho : X \to X^*$, $f \mapsto f^*$ such that

$$\langle u, f^* \rangle = \int_\Omega u \cdot f \, d\mu \qquad (u \in X, f \in X) \qquad . \tag{4.35}$$

Define a.e K_s, $K_t : \Omega \to \mathbf{C}$ by setting

$$K_s(t) \triangleq K_t(s) \triangleq K(s, t) \ \text{ a.e} \qquad ((s, t) \in \Omega) \qquad . \tag{4.36}$$

T is clearly linear. Moreover,

$$|(Tf)(s)| = |\langle K_s, f^* \rangle| \leq \|K_s\|_X \qquad (\|f\|_X < 1) \tag{4.37}$$

(the latter inequality is a Cauchy-Schwarz one). Now apply the Fubini's theorem with $|K|^2$ to obtain

$$\|Tf\|_X^2 \leq \int_\Omega \|K_s\|_X^2 \, d\mu(s) = \|K\|_P^2 < \infty \qquad (\|f\|_X < 1) \qquad . \tag{4.38}$$

(a) is then proved.

To show (b), remark that

$$\int_\Omega a_i(s) \cdot b_i \cdot f \, d\mu \ \in \mathbf{C} \cdot a_i(s) \ \text{ a.e} \qquad (f \in X, s \in \Omega) \qquad . \tag{4.39}$$

It is now clear that T maps any f of X into $\mathbf{C} \cdot a_1 + \cdots + \mathbf{C} \cdot a_n$. We so conclude that $\dim R(T_1) \leq n$.

We now prove (c). The current part refers to Exercise 4.13. X is also a Hilbert space and so contains a Hilbert basis M. Define a.e

$$\begin{aligned} a_b : \Omega &\to \mathbf{C} \\ s &\mapsto (K_s, b) \end{aligned} \tag{4.40}$$

whenever b ranges M. Hence,

$$K_s = \sum_{b \in M} a_b(s) \cdot b \ \text{ a.e} \qquad (s \in \Omega) \qquad . \tag{4.41}$$

Provided any positive scalar ε, there exists a finite subset $S = S(\varepsilon)$ of M such that

$$\|K_s - \sum_{b \in S} a_b(s) \cdot b\|_X < \varepsilon \qquad (s \in \Omega) \qquad . \tag{4.42}$$

Remark that $\sum_{b \in S} a_b \cdot b$ matches the definition of K_1; *cf.*(b): from now on,

$$K_1 \triangleq \sum_{b \in S} a_b \cdot b \qquad . \tag{4.43}$$

It follows from (b) that

$$\dim R(K_1) < \infty \qquad . \tag{4.44}$$

Now turn back to (a), with $K - K_1$ playing the role of K, and so obtain

$$\|T - T_1\| < \varepsilon \mu(\Omega) \leq \infty \qquad . \tag{4.45}$$

For if μ is finite, use (a) of Exercise 4.13 to conclude that T is compact. Assume henceforth that μ is not (necessarily) finite and pick δ in $\mathbf{R}_+$. The simple functions (with finite measure support) form a dense family of an $\mathbf{L}^p$ space ($1 \leq p < \infty$); *cf.*[3.13] of [3]. It then exists a simple function K_δ of $L^2(\mu \times \mu)$ such that

$$(\mu \times \mu)(\{K_\delta \neq 0\}) < \infty, \ \|K - K_\delta\|_P < \delta \qquad . \tag{4.46}$$

Define an operator T_δ in terms of K_δ as T was defined in terms of K, and proceed as in (a) with $T - T_\delta$ instead of T. Then

$$\|T - T_\delta\| < \delta \qquad . \tag{4.47}$$

The key ingredient is that K_δ can be identified with an element of the finite measure space $L^2(\{K_\delta \neq 0\}, \mu \times \mu)$. What we have attempted to approximate T by T_1 can therefore be reiterated (with K_δ playing the role of K) to achieve an approximation $T_{\delta,1}$ of T_δ so that

$$\|T_\delta - T_{\delta,1}\| < \varepsilon \qquad . \tag{4.48}$$

It now follows from (4.47) and (4.48) that

$$\|T - T_{\delta,1}\| \leq \|T - T_\delta\| + \|T_\delta - T_{\delta,1}\| < \varepsilon + \delta \qquad . \tag{4.49}$$

Since ε and δ were arbitrary, the σ-finite case is proved. We now establish (d).

Provided g of X, let E_g be the following equation on X

$$Tf - \lambda f = g \qquad , \tag{4.50}$$

whose solution set is denoted by S_g. Note that S_0 is $\ker(T - \lambda)$ and discard the trivial case $S_0 = X$[1]: each f of X lies in $S_{Tf - \lambda f}$, as some $Tf - \lambda f$'s are nonzero. Some S_g's are then nonempty. Remark that

$$S_g = f + S_0 \qquad (f \in S_g) \tag{4.51}$$

in such case. Furthermore, the equality $\beta = \alpha$ of [4.25] yields

$$(T - \lambda I)(X) \neq X, \text{, } \textit{i.e.}, S_0 \neq \{0\} \qquad . \tag{4.52}$$

So if $T - \lambda I$ is not onto, not only some S_g's are empty, but also $S_0 \neq \{0\}$. Every nonempty S_g (such sets always exist, see above) is then infinite, by (4.51).
Otherwise, $T - \lambda I$ is bijective and every equation E_g has then a unique solution f. The Fredholm alternative is so proved.

Our last step is the description of T^*. Let $S : X \to X$ be such that

$$(Sf)(t) \triangleq \int_\Omega K_t \cdot f \ \text{ a.e} \qquad (f \in X,\ t \in \Omega) \tag{4.53}$$

Proceed as in (a), with S instead of T: S lies in $\mathcal{B}(X)$. Next, we claim that

$$\langle u,\ T^* f^* \rangle = \langle Tu,\ f^* \rangle \tag{4.54}$$

$$= \int_\Omega (Tu) \cdot f \ \ d\mu \tag{4.55}$$

$$= \int_{\Omega^2} K \cdot f \cdot u \ \ d(\mu \times \mu) \tag{4.56}$$

$$= \int_\Omega (Sf) \cdot u \ \ d\mu \tag{4.57}$$

$$= \langle u,\ (Sf)^* \rangle \qquad , \tag{4.58}$$

whenever u and f run through the closed unit ball of X. Since $\|T\|$, $\|T^*\|$ are equal and finite, only exactness of (4.56) is possibly in doubt; the justification below dissipates it. In conclusion,

$$T^* = \rho S \rho^{-1} \qquad . \tag{4.59}$$

Informally,

$$T^* = S \qquad . \tag{4.60}$$

[1] For instance, $X = L^2(\{0\},\ \delta)$ so that $I = \lambda^{-1} T$ is compact.

Justification of (4.56). The current proof shall be complete once we have justified (4.56). To do so, keep u and f as above. Let us introduce

$$\mathrm{A}(\mathrm{s}) \triangleq \int_{\Omega} |\mathrm{K}_{\mathrm{s}}(\mathrm{t}) \cdot \mathrm{u}(\mathrm{t})| \ \mathrm{d}\mu(\mathrm{t}) \ \text{ a.e} \qquad (\mathrm{s} \in \Omega) \qquad , \tag{4.61}$$

to make hold the following Cauchy-Schwarz inequality

$$\mathrm{A}(\mathrm{s}) \leq \|\mathrm{K}_{\mathrm{s}}\|_{\mathrm{X}} \qquad (\mathrm{s} \in \Omega) \qquad . \tag{4.62}$$

Thus,

$$\int_{\Omega^2} |\mathrm{K}(\mathrm{s}, \mathrm{t}) \, \mathrm{u}(\mathrm{t}) \, \mathrm{f}(\mathrm{s})| \ \mathrm{d}\mu(\mathrm{s}) \, \mathrm{d}\mu(\mathrm{t}) = \int_{\Omega} |\mathrm{f}(\mathrm{s})| \, \mathrm{A}(\mathrm{s}) \ \mathrm{d}\mu(\mathrm{s}) \tag{4.63}$$

$$\leq \int_{\Omega} |\mathrm{f}(\mathrm{s})| \, \|\mathrm{K}_{\mathrm{s}}\|_{\mathrm{X}} \ \mathrm{d}\mu(\mathrm{s}) \tag{4.64}$$

$$\leq \left[\int_{\Omega} \|\mathrm{K}_{\mathrm{s}}\|_{\mathrm{X}}^2 \ \mathrm{d}\mu(\mathrm{s}) \right]^{\frac{1}{2}} = \|\mathrm{K}\|_{\mathrm{P}} < \infty \qquad . \tag{4.65}$$

The inequality in (4.65) is a Cauchy-Schwarz one, the following equality follows from the Fubini's theorem. This achieves the proof. □

Chapter 6

Distributions

1 Exercise 1. Test functions are almost polynomial

Suppose f *is a complex continuous function in* $\mathbf{R}^n$*, with compact support. Prove that* $\psi P_j \to f$ *uniformly on* $\mathbf{R}^n$*, for some* $\psi \in \mathcal{D}$ *and for some sequence* $\{P_j\}$ *of polynomials.*

PROOF. According to 1.16, Ω is union of a compact sets sequence $\{K_i\}$ and $\operatorname{supp}(f)$ lies in some $K = K_i$ so that f is embedded in $\mathcal{D}(\Omega)$. We can apply [1.10] to ensure that Ω encloses a compact set $S = \overline{K + B(\varepsilon)}$ for sufficiently small $\varepsilon > 0$.

One easily checks that the Stone-Weierstraß theorem [5.7] can be applied with the subalgebra $\{g \in C(S) : g \text{ is polynomial}\}$ of $C(S)$. There exists a sequence $\{P_j : j \in \mathbf{N}\}$ of $\mathbf{R}[X_1, \dots, X_n]$ such that

$$\sup_S |f - P_j| \underset{j\infty}{\longrightarrow} 0 \qquad . \tag{6.1}$$

By [6.20], the open set $K + B(\varepsilon)$ has a local partition of unity $\{\psi_i\} \subseteq \mathcal{D}(\Omega)$. Moreover, there exists an integer l such that $\psi = \psi_1 + \cdots + \psi_l$ equals 1 on K. Hence

$$\| f - \psi P_j \|_\infty = \| \psi f - \psi P_j \|_\infty = \sup_S |\psi f - \psi P_j| \tag{6.2}$$

$$= \sup_S |f - P_j| \overset{(6.1)}{\underset{j\infty}{\longrightarrow}} 0 \qquad . \tag{6.3}$$

□

We will actually prove more by showing that $\mathcal{D}(\Omega)$ is separable for each nonempty open subset Ω of $\mathbf{R}^n$.

PROOF. The following is split in three parts. The first one is about the above requested result: That was our first part. We now go further by proving the separability of $\mathcal{D}(\Omega)$. To do so, we keep (α, j) in $\mathbf{N}^n \times \mathbf{N}$. Remark that S encloses $\operatorname{supp}(D^\alpha f)$: according to the first part, there exists a sequence $\{P_{\alpha,j} : j \in \mathbf{N}\} \subseteq \mathbf{R}[X_1, \dots, X_n]$ such that

$$\| D^\alpha f - \psi P_{\alpha,j} \|_\infty \underset{j\infty}{\longrightarrow} 0 \qquad . \tag{6.4}$$

Now let m range over $\{1, 2, 3, \dots\}$ and set $W_{m,j}$ in $\mathcal{D}(\Omega)$ as follows

$$D^{-\alpha}\phi \in \mathcal{D}(\Omega) : D^\alpha D^{-\alpha}\phi = \phi \qquad . \tag{6.5}$$

$$W_{m,j}(x) \triangleq D^{-(m,\dots,m)}(\psi P_{(m,\dots,m),j}) \tag{6.6}$$

By (6.4), there exists a natural number $k(m)$ such that

$$\| D^{(m,\dots,m)}(f - W_{m,j}) \|_\infty < 1/m \qquad (j \geq k(m)) \qquad . \tag{6.7}$$

Assume without loss of generality that S has diameter 1so that (6.7) yields

$$\| D^\lambda (f - W_{m,k(m)}) \|_\infty < 1/m \qquad (|\lambda| \leq m) \qquad , \tag{6.8}$$

by the mean value theorem. In other words (remark that $\mathrm{supp}(f - W_{m,k(m)}) \subseteq S$),

$$f - W_{m,k(m)} \in U_m \triangleq \{\phi \in \mathcal{D}_S : \|\phi\|_m < 1/m\} \supseteq U_{m+1} \supseteq \cdots \qquad (m = 1, 2, 3, \ldots) \qquad . \tag{6.9}$$

Pick W in β (see (b) of [6.3]): $W \cap \mathcal{D}_S$ contains a neighborhood of 0. Hence W contains some U_m, for m sufficiently large. Thus

$$W_{m,k(m)} \underset{m\infty}{\longrightarrow} f \qquad (\text{in } \mathcal{D}(\Omega)) \qquad . \tag{6.10}$$

We have so established that the $W_{m,k(m)}$'s family is dense in $\mathcal{D}(\Omega)$. We now aim to disclose a countable set $\tilde{W}$ that has the same property.

Choose δ in $\mathbf{R}_+$ and fetch any $W_{m,k(m)}$. Let X be $(X_1, \ldots, X_n)$ and express $P_{(m,\ldots,m),k(m)}$ as

$$P(X) = \sum_{|\gamma| \le d} p_\gamma \cdot X^\gamma \qquad . \tag{6.11}$$

Since $\bar{\mathbf{Q}} = \mathbf{R}$, $\mathbf{Q}[X]$ hosts some $\mathbf{Q}(X) = \sum_{|\gamma| \le d} q_\gamma \cdot X^\gamma$ such that $|p_\gamma - q_\gamma| < \delta$ for all γ. Thus,

$$|P(x) - Q(x)| \le \sum_{|\gamma| \le d} |p_\gamma - q_\gamma| \, |x|^{|\gamma|} \le \delta \sum_{l \le d} \mathtt{C}(\mathtt{l} + \mathtt{n} - \mathtt{1}, \mathtt{n} - \mathtt{1}) \, \|x\|_\infty^l \qquad (x \in \mathbf{R}^n) \qquad . \tag{6.12}$$

Since S is bounded, we so obtain

$$\|\psi(P - Q)\|_\infty \in O(\delta) \qquad . \tag{6.13}$$

Now define $\tilde{W}_m$ in terms of Q as $W_{m,k(m)}$ was defined in terms of P, and consider the integrations made in (6.6): each $D^\lambda \tilde{W}_m$ ($|\lambda| \le m$) can be obtained from some of them. So (6.13) yields

$$\| D^\lambda (W_{m,k(m)} - \tilde{W}_m) \|_\infty \in O(\delta) \qquad (|\lambda| \le m) \qquad . \tag{6.14}$$

To be more specific, these λ's only exist in finite amount, so the big O can be assumed to be the same for all them. Since δ was arbitrary, combining (6.10) with (6.14) establishes the density of the all $\tilde{W}_m$'s family $\tilde{W}$.

Furthermore, each member of $\tilde{W}$ is only made of two ingredients: ψ and a polynomial of $\mathbf{Q}[X]$. The mapping ψ is attached to some K_i and $\mathbf{Q}[X]$ inherits countableness from $\mathbf{Q}$. Note that the "integrations packs" of (6.6) only exist in countable amount. Our $\tilde{W}$ is then countable. □

6 Exercise 6. Around the supports of some distributions

(a) Suppose that $c_m = \exp\{-(m!)!\}$, $m = 0, 1, 2, \ldots$. *Does the series*

$$\sum_{m=0}^{\infty} c_m \, D^m \, \phi(0)$$

converges for every $\phi \in C^\infty(R)$*?*

(b) Let Ω *be open in* $\mathbf{R}^n$*, suppose* $\Lambda_i \in \mathcal{D}'(\Omega)$*, and suppose that all* Λ_i *have their supports in some fixed compact* $K \subseteq \Omega$*. Prove that the sequence* $\{\Lambda_i\}$ *cannot converge in* $\mathcal{D}'(\Omega)$ *unless the orders of the* Λ_j *are bounded. Hint: Use the Banach-Steinhaus theorem.*

(c) Can the assumption about the supports be dropped in (b)?

PROOF. The answer is: no. To establish this assertion, we first assume, to reach a contradiction, that the above series converges for every smooth $\phi : \mathbf{R} \to \mathbf{C}$.

The sequence $\{c_m \, D^m \, \phi(0)\}$ so converges to 0. Nevertheless, it is proved in [1.46] that $C^\infty(\Omega)$ is not locally bounded. In other words, it is always possible to excavate a ϕ for which the magnitude of the m-th derivative at 0 is

as large as we please[1], *e.g.*, greater than $1/c_m$. A desired contradiction is then reached. We now prove (b), again by contradiction.

To do so we assume $\{\Lambda_j\}$ to converge to some Λ of $\mathcal{D}'(\Omega)$ and we let Q run through the compact sets of Ω. Next, we define

$$S(T, Q) \triangleq \{N \in \mathbf{N},\ \exists C \in \mathbf{R}_+ :\ |T\phi| \le C\|\phi\|_N \text{ for all } \phi \text{ of } \mathcal{D}_Q\} \qquad (T \in \mathcal{D}(\Omega)) \qquad . \tag{6.15}$$

Such subset of $\mathbf{N}$ has a minimum $\omega(T, Q)$. The following value

$$\omega(T) \triangleq \max\{\omega(T, Q) :\ Q \subseteq \Omega,\ Q \text{ compact }\} \le \infty \tag{6.16}$$

is then the order of T. Assume, to reach a contradiction, that, after passage to a subsequence,

$$\omega(\Lambda_j, Q_j) = j \qquad (j = 1, 2, 3, \ldots) \tag{6.17}$$

for some compact $Q = Q_j$. By (a) of [6.24], Q_j cuts $\mathrm{supp}\Lambda_j$, say in p_j. Since K encloses $\mathrm{supp}\Lambda_j$, $\{p_j\}$ tends, after passage to a subsequence, to some p of K. Choose a positive scalar r so that

$$\overline{B}(p, r) \triangleq \{x \in \mathbf{R}^n :\ |x - p| \le r\} \subseteq \Omega \qquad . \tag{6.18}$$

Such closed ball $\overline{B}(p, r)$ is a compact subset of Ω. By (b) of [6.5] (which refers to [1.46]) $\mathcal{D}_{\overline{B}(p, r)}$ is then a Fréchet space. It now follows from [2.6] that $\{\Lambda_j\}$ is equicontinuous on $\mathcal{D}_{\overline{B}(p, r)}$. There exists[2] a nonnegative integer N such that

$$|\Lambda\phi| \le C\|\phi\|_N \qquad (\phi \in \mathcal{D}_{\overline{B}(p, r)}) \tag{6.19}$$

for some positive constant C. On the other hand, $\overline{B}(p, r)$ contains almost all the p_j's. Hence

$$|\Lambda_N \phi| > C\|\phi\|_N \tag{6.20}$$

for some ϕ of $\mathcal{D}_{\overline{B}(p,r)}$. (b) is then established.

To prove (c), we introduce a sequence $\{x_m :\ m \in \mathbf{Z}\}$ of Ω that has no limit point. Let $\{\alpha_m :\ m \in \mathbf{Z}\}$ be in $\mathbf{N}$ and so define[3]

$$\begin{aligned} \Lambda :\ \mathcal{D}(\Omega) &\to \mathbf{C} \qquad . \\ \phi &\mapsto \sum_{m=-\infty}^{\infty} D^{\alpha_m}\phi(x_m) \end{aligned} \tag{6.21}$$

Λ belongs to $\mathcal{D}'(\Omega)$, since $\{x_m\}$ has no limit point. Next, we easily check that

$$\begin{aligned} \Lambda_j :\ \mathcal{D}(\Omega) &\to \mathbf{C} \qquad (j \in \mathbf{N}) \\ \phi &\mapsto \sum_{|m|\le j} D^{\alpha_m}\phi(x_m) \end{aligned} \tag{6.22}$$

is also a distribution and that $\{\Lambda_j\}$ tends to Λ in $\mathcal{D}'(\Omega)$. Nevertheless, no Λ_j's can have common support because $\{x_m\}$ has no limit point. Our assumption can therefore be dropped. □

9 Exercise 9. Convergence in $\mathcal{D}$ vs. convergence in $\mathcal{D}'$

(a) Prove that a set $E \subseteq \mathcal{D}(\Omega)$ *is bounded if and only if*

$$\sup\{|\Lambda\phi| :\ \phi \in E\} < \infty$$

for every $\Lambda \in \mathcal{D}(\Omega)$.

[1] indeed [1.46] provides sufficient tools for constructive proof of this.

[2] For more details, see Exercise 2.3.

[3] As $\Omega = \mathbf{R}$, the case $\alpha_m = m$ is the "counterpart" of the series of (a) and the case $(x_m, \alpha_m) = (m, 0)$ is the Dirac comb.

(b) Suppose $\{\phi_j\}$ *is a sequence in* $\mathcal{D}(\Omega)$ *such that* $\{\Lambda\phi_j\}$ *is a bounded sequence of numbers, for every* $\Lambda \in \mathcal{D}'(\Omega)$. *Prove that some subsequence of* $\{\phi_j\}$ *converges, in the topology of* $\mathcal{D}(\Omega)$.

(c) Suppose $\{\Lambda_j\}$ *is a sequence in* $\mathcal{D}'(\Omega)$ *such that* $\{\Lambda_j\,\phi\}$ *is bounded, for every* $\phi \in \mathcal{D}(\Omega)$. *Prove that some subsequence of* $\{\Lambda_j\}$ *converges in* $\mathcal{D}'(\Omega)$ *and that the convergence is uniform on every bounded subset of* $\mathcal{D}(\Omega)$. *Hint: By the Banach-Steinhaus theorem, the restrictions of the* Λ_j *to* $\mathcal{D}_K$ *are equicontinuous. Apply Ascoli's theorem.*

PROOF. Since $\mathcal{D}(\Omega)$ is a locally convex space (see (b) of [6.4]), [3.18] states that E is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that $E = \{\phi_j : j \in \mathbf{N}\}$ is bounded: so is $\overline{E}$. By (c) of [6.5], there exists some $\mathcal{D}_K$ that contains $\overline{E}$. Since $\mathcal{D}_K$ has the Heine-Borel property (see [1.46]), $\overline{E}$ is τ_K-compact. Apply [A4] with the metrizable space $\mathcal{D}_K$ (see [1.46]) to conclude that $\overline{E}$ has a τ_K limit point. It then follows from (b) of [6.5] that (b) holds. □

Annex

A.1 Vector spaces

Lemma A.1 Vector subspaces as convex and balanced sets.

Given a vector space X, the following are equivalent for any nonempty $S \subseteq X$:

(a) S is a vector subspace of X,

(b) S is balanced, and $S + S \subseteq S$,

(c) S is convex and balanced, and $\lambda S \subseteq S$ for all $\lambda > 1$.

PROOF. It suffices to show that (a) → (b), (b) → (c), and (c) → (a). Clearly, (a) implies that $S + S \subseteq S$ and that S is balanced, which is (b). Next, let us work with (b) alone. By balancedness,

$$(1-t)\,S + t\,S \subseteq S + S \subseteq S \qquad (0 \le t \le 1), \tag{A.1}$$

which establishes the convexity of S. Furthermore, a straightforward induction on $n = 2, 3, 4, \ldots$ shows that

$$n\,S \subseteq (n-1)\,S + S \subseteq S + S \subseteq S. \tag{A.2}$$

Now choose $\lambda > 1$. The special case $n = \lceil \lambda \rceil$ yields

$$S \subseteq \lambda\,S \subseteq \lceil \lambda \rceil\,S \subseteq S, \tag{A.3}$$

since S is balanced. Therefore,

$$S = \lambda\,S, \tag{A.4}$$

which completes the proof of (b) → (c). Finally, assume (c) only and consider all $\lambda = 2 + |\alpha|$ ($\alpha \in \mathbf{C}$). Hence

$$\alpha\,S + \alpha'\,S \subseteq \lambda\,S + \lambda'\,S \qquad \text{(by balancedness)} \tag{A.5}$$

$$\subseteq S + S \qquad \text{(by the assumption that } \lambda S \subseteq S \text{ for all } \lambda > 1) \tag{A.6}$$

$$\subseteq 2\,S \qquad \text{(by convexity; see (1.8))} \tag{A.7}$$

$$\subseteq S. \qquad \text{(by the assumption that } \lambda S \subseteq S \text{ for all } \lambda > 1) \tag{A.8}$$

In summary, $\alpha\,S + \alpha'\,S \subseteq S$ holds for all scalars α and α'. This proves (c) → (a) and completes the proof. □

From now on, Y has finite dimension $n > 0$: the trivial 0-dimensional case is excluded. In the context of topological vector spaces, as defined in [1.6, 1.12] of [4], we now establish the fundamental result:

In finite dimension, all norms are equivalent.

By "equivalent", we mean that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a finite-dimensional vector space Y have the same order of magnitude: There exist positive constants C and C′ such that, for all $y \in Y$,

$$\|y\|' \le C\,\|y\|, \tag{A.9}$$

$$\|y\| \le C'\|y\|'. \tag{A.10}$$

This expresses in terms of norms a fundamental *topological* equivalence between all vector spaces of same dimension n. Theorem [A.3] makes this equivalence explicit. It uses the following definitions:

(i) Given a vector space U, the group Aut(U) consists of all isomorphisms of U onto itself.

(ii) Given vector spaces U and V, Iso(U, V) is the family of all isomorphisms of U onto V.

(iii) Given topological vector spaces U and V, GL(U, V) is the subset of all $\Lambda \in \mathrm{Iso}(U, V)$ that are homeomorphisms. Elements of GL(U, V) are known as the *topological isomorphisms* of U onto V.

(iv) GL_n is the *general linear group* $\mathrm{GL}(\mathbf{C}^n, \mathbf{C}^n)$, where $\mathbf{C}^n$ is endowed with the product topology.

We start with Lemma [A.2], which assigns to each isomorphism $\mathbf{C}^n \to Y$ a basis of Y.

Lemma A.2.

Let Y be a complex vector space of dimension n. The isomorphisms of $\mathbf{C}^n$ onto Y exist: they are exactly the linear maps f that solve the systems

$$E_u : \begin{cases} f(e_1) & = u_1 \\ \vdots & \\ f(e_n) & = u_n \end{cases}$$

when the lists $u_1, \dots, u_n$ describe the bases of Y. More precisely:

(a) Each E_u has a unique solution $f_u : \mathbf{C}^n \to Y$ that is a linear isomorphism, and, conversely,

(b) to each isomorphism $f : \mathbf{C}^n \to Y$ corresponds a unique system E_u that encodes $f = f_u$.

PROOF. Let $u_1, \dots, u_n$ describe any basis of Y. Then then each E_u defines a function that extends to a unique linear mapping $f = f_u$, where

$$f\left(\sum_{k=1}^{n} \alpha_k e_k\right) = \sum_{k=1}^{n} \alpha_k u_k \qquad (\alpha_k \in \mathbf{C}). \tag{A.11}$$

Note that f is one-to-one, since the whole list $u_1, \dots, u_n$ is linearly independent. The whole image $f(\mathbf{C}^n)$ has then dimension n. Hence $f(\mathbf{C}^n) = Y$, which completes the proof of (a). Conversely, if $f : \mathbf{C}^n \to Y$ is an isomorphism, then $v_1 = f(e_1), \dots, v_n = f(e_n)$ describes a basis of Y. Equality (A.11) applied to this f shows that the system E_v carries enough information to retrieve f. □

We now give the central result: all vector spaces Y of same dimension n are topologically equivalent, in the sense that the only way to endow Y with a vector space topology is to copy the product topology of C^n onto Y with any isomorphism $f : \mathbf{C}^n \to Y$. The normed point of view is addressed in Corollary [A.5].

Theorem A.3 Topology of a finite-dimensional vector space.

If Y is a complex vector space of finite dimension n, then Y admits a unique vector topology, which is the product topology of $\mathbf{C}^n$, transferred by *any* isomorphism $f : \mathbf{C}^n \to Y$. This topology is normable, and Y is locally compact, locally convex, and locally bounded.

Moreover, if W is a complex topological vector space, then the following statements are equivalent:

(a) $\dim W = n$.

(b) W has a vector topology, and $\mathrm{GL}(\mathbf{C}^n, W) \neq \emptyset$.

(c) For all $g \in \mathrm{GL}(\mathbf{C}^n, W)$ (they exist), $\mathrm{GL}(Y, W) = g \circ \mathrm{GL}_n \circ f^{-1}$ entirely fills Iso(Y, W).

PROOF. Let υ be the product topology of $\mathbf{C}^n$. Choose an isomorphism $f_0 : \mathbf{C}^n \to Y$ (existence was proved with Lemma [A.2]), then equip Y with the vector topology

$$\tau_0 = \{f_0(\Omega) : \Omega \in \upsilon\}. \tag{A.12}$$

Indeed, the bijectivity of f_0 implies the following four properties:

(i) $\bigcup_{U\in\Gamma} f_0(U) = f_0\Big(\bigcup_{U\in\Gamma} U\Big) \in \tau_0; Y \in \tau_0 \qquad \Gamma \subseteq \upsilon.$

(ii) $f_0(U) \cap f_0(U') = f_0(U \cap U'); \emptyset \in \tau_0.$

(iii) τ_0 is translation invariant, and $\{f_0(0)\}$ is closed.

(iv) The transfer by f_0 of addition and scalar multiplication preserves continuity.

Now apply [1.21] of [4] in the special case where the ambient space X is Y itself, equipped with a vector space topology τ, *e.g.*, τ_0. So, every isomorphism $f : \mathbf{C}^n \to Y$ is also a homeomorphism. In summary,

$$\mathrm{Iso}(\mathbf{C}^n, Y) = \mathrm{GL}(\mathbf{C}^n, Y) \neq \emptyset. \tag{A.13}$$

Thus, every $f \in \mathrm{Iso}(\mathbf{C}^n, Y)$ transports the product topology of $\mathbf{C}^n$ onto Y. A vector space topology τ for Y is then necessarily of the form

$$\big\{f(\Omega) : \Omega \in \upsilon\big\} = \tau \qquad \Big(f \in \mathrm{Iso}(\mathbf{C}^n, Y)\Big). \tag{A.14}$$

The special case $f = f_0$ yields $\tau = \tau_0$. We have just proved that

$$\tau_0 \text{ is the only admissible vector topology of Y.} \tag{A.15}$$

Our first statement is then established. The special case $Y = \mathbf{C}^n$ is that every isomorphism $f : \mathbf{C}^n \to \mathbf{C}^n$ is, more specifically, a topological isomorphism. So,

$$\mathrm{Aut}(\mathbf{C}^n) = \mathrm{GL}_n\,. \tag{A.16}$$

We now return to the general Y.

Definition (A.12) implies local convexity and local boundedness. Consequently, Y is normable; see [1.39] of [4]. Constructively, the following formula

$$\|y\| \triangleq \big\| f_0^{-1}(y) \big\|_2 \qquad (y \in Y) \tag{A.17}$$

introduces a norm $\|\cdot\|$ on Y. Another strong property is that Y is locally compact, since the closed unit ball of $\mathbf{C}^n$ is compact and f_0 is a homeomorphism.

Now let W be a complex vector space. The preceding argument (with W in place of Y) shows that, if W has dimension n, then

- W is a topological vector space,
- there exists a topological isomorphism $g : \mathbf{C}^n \to W$.

Hence (a) $\to$ (b).

Now assume (b) only. We easily check that the diagram [A.1] below is commutative and that

$$\mathrm{Iso}(Y, W) = g \circ \mathrm{Aut}(\mathbf{C}^n) \circ f^{-1} \qquad \Big(f \in \mathrm{Iso}(\mathbf{C}^n, Y)\Big). \tag{A.18}$$

Moreover, combining (A.13) with (A.16) shows that the (then nonempty) set $\mathrm{Iso}(Y, W)$ consists of all topological isomorphisms of Y onto W. This completes the proof of (b) $\to$ (c). Finally, (c) alone implies that there exists an isomorphism $g : \mathbf{C}^n \to W$. The list $g(e_1), \ldots, g(e_n)$ then describes a basis of W, which proves (a). This completes the proof. □

When two complex vector spaces Y and W have the same finite dimension $n > 0$, Theorem [A.3] shows that they are linearly homeomorphic and that both are normable. We now give its normed counterpart, which expresses topological equivalence in terms of continuity bounds.

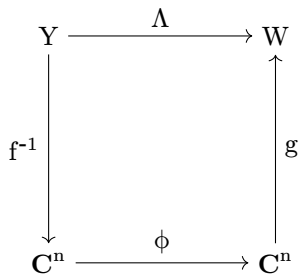


Figure A.1. The bijection $\mathrm{Aut}(\mathbf{C}^n) \to \mathrm{Iso}(Y, W), \phi \mapsto \Lambda = g \circ \phi \circ f^{-1}$.

Theorem A.4 Equivalence of norms in a finite-dimensional space.

If Y is a vector space of finite dimension n, then any two norms $\|\cdot\|$ and $\|\cdot\|'$ on Y are *equivalent*. More specifically, there exist two positive constants C and C′ such that

$$\frac{1}{C'}\,\|y\| \le \|y\|' \le C\,\|y\| \qquad (y \in Y). \tag{A.19}$$

The values of C and C′ may depend on the choice of the norms.

PROOF. Both norms define the same topology; see (A.15). Therefore, the identity $I : \big(Y, \|\cdot\|\big) \to \big(Y, \|\cdot\|'\big)$ is continuous. Equivalently, given $M > 0$, there exists a radius $r = r(M)$ such that

$$\|y_1 - y_0\|' < M \qquad \big(\|y_1 - y_0\| < r\big). \tag{A.20}$$

With these settings, the reverse triangle inequality $\big|\,\|y_1\|' - \|y_0\|'\,\big| \le \|y_1 - y_0\|'$ yields

$$\big|\,\|y_1\|' - \|y_0\|'\,\big| < M. \tag{A.21}$$

Consequently, $\|\cdot\|'$ is uniformly continuous. Next, take $M = 1$. Let R be the supremum taken over all admissible $r(1)$. This R is finite because $R = \infty$ would force $1 > \|ky\|' \ge 1$ for $y \ne 0$ and $k = \lceil 1/\|y\|' \rceil$. For arbitrary $\alpha > 1$, fix

$$y_1(y) \triangleq \frac{R/\alpha}{\|y\| + (R/\alpha)(\alpha - 1)}\,y \qquad (y \in Y), \tag{A.22}$$

so that

$$y = \left[\frac{\alpha}{R}\,\|y\| + \alpha - 1\right] y_1(y) \tag{A.23}$$

and

$$\|y_1(y)\| < \frac{R}{\alpha} < R. \tag{A.24}$$

Inequality (A.20), with $y_0 = 0$ and $y_1 = y_1(y)$, yields

$$\|y_1(y)\|' < 1. \tag{A.25}$$

Next, combine this with (A.23) to obtain

$$\|y\|' < \frac{\alpha}{R}\,\|y\| + \alpha - 1 \tag{A.26}$$

for all y. Since α was arbitrary, it follows that

$$\|y\|' \le \frac{1}{R}\,\|y\| \qquad (y \in Y). \tag{A.27}$$

Therefore, $C = 1/R$ is the optimal constant. Exchanging the roles of the two norms shows that $\|\cdot\|$ is uniformly continuous. Similarly, there exists a positive C' such that

$$\|y\| \le C' \|y\|' \qquad (y \in Y). \tag{A.28}$$

Combining (A.28) with (A.27) establishes (A.19). Table 1 lists optimal C for $I : \left(\mathbf{C}^n, \|\cdot\|_i\right) \to \left(\mathbf{C}^n, \|\cdot\|_j\right)$ when $\|\cdot\|_i$ and $\|\cdot\|_j$ are the standard norms $1, 2, \infty$. They are computed independently of this proof. This shows that the optimal constants, at least in these cases, depend on the choice of the norms. □

Corollary A.5 Norms on finite-dimensional vector spaces.

Let Y, W be vector spaces of finite dimension n, and let $\|\cdot\|_Y$ (respectively $\|\cdot\|_W$) be a norm on Y (respectively W). If h is an isomorphism of Y onto W, then $\|h(\cdot)\|_W$ is a norm on Y. Furthermore, $\|h(\cdot)\|_W$ is uniformly continuous; there exist positive constants C and C' such that

$$\frac{1}{C'}\|y\|_Y \le \|h(y)\|_W \le C\|y\|_Y \qquad (y \in Y). \tag{A.29}$$

PROOF. First, we note that $\|h(y)\|_W = 0$ **iff** $y = 0$, because the linear map h is one-to-one. The linearity of h also implies that $\|h(\cdot)\|_W$ satisfies all other norm axioms. Theorem [A.4] now applies, with $\|h(\cdot)\|_W$ in the role of $\|\cdot\|'$. This completes the proof. □

A.2 Mean value and bounded derivatives

Lemma A.6 A mean value inequality for higher-order derivatives.

Let $[a, b] \subseteq \mathbf{R}$. If $\phi \in \mathcal{D}_{[a,b]}$, then

$$\left\| D^N \phi \right\|_\infty \le \left\| D^{N+p} \phi \right\|_\infty \cdot \frac{1}{p!}\left[\frac{b-a}{2}\right]^p \tag{A.30}$$

for all pairs $(N, p) \in \mathbf{N}^2$.

PROOF. Let x_0 be a or b, so that $\phi(x_0) = 0$. We show by induction that the following property $\mathbf{Ineq}_p(\phi)$

$$|\phi(x)| \le \|D^p \phi\|_\infty \frac{|x - x_0|^p}{p!} \qquad (x \in \mathbf{R}) \tag{A.31}$$

holds for every (p, ϕ). First, note that $\mathbf{Ineq}_0(\phi)$ is trivial. Next, assume that, for a fixed p, every $\mathbf{Ineq}_p(\phi)$ is true. Clearly, under this assumption, $\mathbf{Ineq}_p(D\phi)$ is true as well, since $D\phi \in \mathcal{D}_{[a,b]}$. In this context, for all $x \in \mathbf{R}$,

$$|\phi(x)| \le \left| \int_{x_0}^{x} |D\phi(u)|\, du \right| \overset{\mathbf{Ineq}_p(D\phi)}{\le} \left\| D^{p+1}\phi \right\|_\infty \left| \int_{x_0}^{x} \frac{|u - x_0|^p}{p!}\, du \right| = \left\| D^{p+1}\phi \right\|_\infty \cdot \frac{|x - x_0|^{p+1}}{(p+1)!}. \tag{A.32}$$

This establishes $\mathbf{Ineq}_{p+1}(\phi)$, and thus completes the induction. The inequality (A.31) now gives

$$|\phi(x)| \le \|D^p \phi\|_\infty \cdot \frac{1}{p!} \cdot \min\left\{|x-a|^p, |x-b|^p\right\} \tag{A.33}$$

$$\le \|D^p \phi\|_\infty \cdot \frac{1}{p!}\left[\frac{b-a}{2}\right]^p \qquad \left(x \in [a, b]\right). \tag{A.34}$$

Hence

$$\|\phi\|_\infty \le \|D^p \phi\|_\infty \frac{1}{p!}\left[\frac{b-a}{2}\right]^p. \tag{A.35}$$

Inequality (A.30) follows when we substitute $D^N \phi$ for ϕ, which completes the proof. □

Lemma A.7 Higher derivatives cannot be uniformly bounded by lower derivatives.

Let $a < b$ in $\mathbf{R}$. Then, for any pair $(N, p) \in \mathbf{N}^2$, there is no positive constant C such that

$$\| D^{N+p+1} \phi \|_\infty \leq C \| D^N \phi \|_\infty \qquad \big(\phi \in \mathcal{D}_{[a,b]}\big). \tag{A.36}$$

Since $\phi|_{[a,b]} \in C^\infty\big([a,b]\big)$ when $\phi \in \mathcal{D}_{[a,b]}$, the result carries over to $C^\infty\big([a,b]\big)$.

PROOF. Up to an affine change of variable, we may assume that $[a,b] = [-1,1]$. We know from the construction described in [1.46] of [4] that there exists $g \in C^\infty(\mathbf{R})$ such that

(a) $0 \leq g \leq 1$,

(b) $g|_{(-\infty,1/2[} = 0$,

(c) $g|_{]1,\infty)} = 1$,

(d) $\| D^N g \|_\infty \leq \frac{2^N}{\delta_1 \cdots \delta_N} = 2^{\frac{N^2+7N}{2}}$, with $\delta_k = \frac{1}{4} 2^{-k}$ $(k = 1, \dots, N)$.

Define

$$\begin{aligned} \phi_0 : \mathbf{R} &\to [0,1] \\ t &\mapsto 1 - g\big(|t|\big), \end{aligned} \tag{A.37}$$

so that

(e) $\phi_0 \in \mathcal{D}_{[-1,1]}$,

(f) $\phi_0|_{[-1/2,1/2]} = 1$,

(g) $0 \leq \phi_0 \leq 1$,

(h) $\| D^N \phi_0 \|_\infty \leq 2^{\frac{N^2+7N}{2}}$ for all N.

Items (d) and (h) yield a constant M_N that depends only on N; see (A.41). To each *angular frequency* $\omega \geq 2\pi$ corresponds the function

$$\begin{aligned} \phi_\omega : \mathbf{R} &\to [-1,1] \\ t &\mapsto \phi_0(t) \sin(\omega t). \end{aligned} \tag{A.38}$$

We now turn to the proof proper. Note that $\phi_\omega \in \mathcal{D}_{[-1,1]}$, since $\phi_0 \in \mathcal{D}_{[-1,1]}$ and $t \mapsto \sin(\omega t)$ is smooth. Assume, to reach a contradiction, that for some pair (N, p) there exists a positive constant C satisfying (A.36). We look for an angular frequency $\omega_\perp$ that yields a contradiction. To this end, Lemma [A.6], with $m = 0, \dots, N$ in place of p and $N - m$ in place of N, gives

$$\sum_{m=0}^{N} \| D^{N-m} \phi_0 \|_\infty \leq \sum_{m=0}^{N} \frac{1}{m!} \| D^N \phi_0 \|_\infty \overset{(h)}{<} 2^{\frac{N^2+7N}{2}} e. \tag{A.39}$$

Moreover, $|D^m \sin(\omega t)| \leq \omega^m \leq \omega^N$ for $m = 0, \dots, N$ and $t \in \mathbf{R}$, since $\omega > 1$. Thus, by the general Leibniz rule,

$$\| D^N \phi_\omega \|_\infty \leq \sum_{m=0}^{N} \mathtt{C}(N, m) \| D^{N-m} \phi_0 \|_\infty \omega^N. \tag{A.40}$$

The binomial identity $(1+1)^N = \sum \mathtt{C}(N, m)$ implies $\mathtt{C}(N, m) \leq 2^N$. Therefore, combining (A.40) with (A.39) gives

$$\| D^N \phi_\omega \|_\infty \leq 2^N \sum_{m=0}^{N} \| D^{N-m} \phi_0 \|_\infty \omega^N < \underbrace{2^{\frac{N^2+9N}{2}} e}_{\triangleq M_N} \omega^N. \tag{A.41}$$

As announced, the factor M_N in (A.41) depends only on N. On the other hand, $t \mapsto \sin(\omega t)$ runs through a complete period over $[-1/2, 1/2]$, since $\omega \geq 2\pi$. Consequently, the maximum of $t \mapsto D^{N+p+1} \sin(\omega t)$, which is ω^{N+p+1}, is attained during this period. Since ϕ_0 meanwhile plateaus at 1, $\phi_\omega(t) = \sin(\omega t)$ when $t \in [-1/2, 1/2]$. Thus,

$$\| D^{N+p+1} \phi_\omega \|_\infty \geq \max_{[-1/2,1/2]} D^{N+p+1} \phi_\omega = \omega^{N+p+1} \geq \omega^{N+1}. \tag{A.42}$$

Combining this estimate with the preceding ones, we obtain

$$\omega^{N+1} \overset{(A.42)}{\leq} \| D^{N+p+1} \phi_\omega \|_\infty \overset{(A.36)}{\leq} C \| D^N \phi_\omega \|_\infty \overset{(A.41)}{<} C M_N \omega^N. \tag{A.43}$$

Dividing both sides by ω^N gives

$$\omega < C M_N. \tag{A.44}$$

Finally, the angular frequency

$$\omega_\perp \triangleq 2\pi + C M_N \tag{A.45}$$

violates the latter inequality, which completes the proof by contradiction. □

The proof of Lemma [A.7] relies on oscillation. A bounded function, however, does not need to oscillate to have a large derivative on a small interval: the following smooth example ϕ keeps a constant positive derivative around 0.

Let $\rho \in C^\infty(\mathbf{R})$ be 1 on $(-\infty, 0]$, 0 on $[1, \infty)$, and strictly decreasing on $]0, 1[$. For instance, choose

$$\rho(t) \triangleq 1 - \frac{\exp\left(-\frac{1}{t}\right)}{\exp\left(-\frac{1}{t}\right) + \exp\left(-\frac{1}{1-t}\right)} \tag{A.46}$$

for all $0 < t < 1$. Inspired by signal processing, we introduce three positive quantities: the rate (ω), the maximum slope (A), and the delay (τ). We now define ϕ to be the solution of the following system

$$\begin{aligned} \phi(0) &= 0 \\ d\phi &= A\rho\Big(\omega\big(|t| - \tau\big)\Big)\, dt. \end{aligned} \tag{A.47}$$

Equivalently, ϕ is odd and, for $s > 0$,

$$\phi(s) = A \int_0^s \rho\big(\omega(t-\tau)\big)\, dt = A \min(s, \tau) + A \int_\tau^{\max(s,\tau)} \rho\big(\omega(t-\tau)\big)\, dt. \tag{A.48}$$

Since $\rho(u) + \rho(1-u) = 1$ for $0 < u < 1$, we have $\int_0^1 \rho(u)\, du = 1/2$. Hence

$$\| \phi \|_\infty = \phi(\tau + 1/\omega) \tag{A.49}$$

$$= \tau A + \frac{A}{2\omega}. \tag{A.50}$$

Setting $A = \omega = 1/\tau$ proves the claim. Indeed, $D\phi|_{[-\tau,\tau]} = A \to \infty$ as $\tau \to 0$. In contrast, $\| \phi \|_\infty = 3/2$.

A.3 Dirac's impulse, a physicist's detour

Consider a physical example: a particle colliding with a surface that absorbs a unit of energy at impact time $t = 0$. We start with H as the *Heaviside step function* $t \mapsto [t \geq 0]$, so that H(t) indicates whether the particle has contributed its energy by time t. This formalism expresses that

1. The energy is transferred by an instantaneous jump at time $t = 0$.
2. The energy is conserved over time.

Heuristically, we write

$$H'(t) = \begin{cases} \infty & (t = 0) \\ 0 & (t \neq 0) \end{cases}, \tag{A.51}$$

$$\int_{\mathbf{R}} H' = 1, \tag{A.52}$$

These properties cannot coexist in standard calculus. Nevertheless, the informal density H' describes the *Dirac* δ *function*. When identified with a positive Borel measure, δ has total mass 1 because

$$\int_{\mathbf{R}} d\delta = \int_{\mathbf{R}} H' = \Big[H \Big]_{-\infty}^{\infty} = 1. \tag{A.53}$$

Physically, integrating over time recaptures all the energy. Let W be the observation window, which is adjusted so that either $0 \in W^\circ$ or $0 \notin \overline{W}$. Next, consider all smooth functions ϕ with (nonempty) compact support in W° as test signals. In this formalism, the integral $\int_{\mathbf{R}} \phi \, d\delta$ represents the detector's response to the collision. Lebesgue's dominated convergence theorem ensures

$$\sup_{0 \leq \phi \leq 1} \int_{W^\circ} \phi \, d\delta = \int_{W^\circ} d\delta = \big[0 \in W^\circ \big]. \tag{A.54}$$

We now make the model rigorous by eliminating the heuristic H', as follows:

$$\int_{\mathbf{R}} \phi \, d\delta = \int_{\mathbf{R}} \phi \, H' \qquad \Big(\text{generalization of (A.54)}\Big) \tag{A.55}$$

$$= \Big[H\phi \Big]_{-\infty}^{\infty} - \int_{\mathbf{R}} H\phi' \qquad \text{(integration by parts)} \tag{A.56}$$

$$= - \int_{\mathbf{R}} H\phi' \qquad \text{(supp}(\phi)\text{ is compact)} \tag{A.57}$$

$$= \phi(0). \tag{A.58}$$

The key point is that the right-hand side in (A.58) is valid in standard calculus. Moreover, we obtain all filtered responses as the evaluation functional $\phi \mapsto \phi(0)$. This motivates the following definitions:

$$\Lambda_H(\phi) \triangleq \int_{\mathbf{R}} H\phi, \qquad \text{(expresses H)} \tag{A.59}$$

$$\Lambda'_H(\phi) \triangleq - \int_{\mathbf{R}} H\phi', \qquad \text{(the } \textit{weak derivative} \text{ of H)} \tag{A.60}$$

and

$$\delta(\phi) \triangleq \phi(0) = \Lambda'_H(\phi). \qquad \text{(impulse at 0: now } \delta \text{ has a rigorous definition)} \tag{A.61}$$

The functional $\delta : \phi \mapsto \phi(0)$ represents an instantaneous energy injection at $t = 0$. Extended to all $\phi \in C_c(\mathbf{R})$, δ is a positive Radon measure of norm/total variation $\|\delta\| = 1$ and support $\{0\}$. In the sense of distribution theory, δ is a (tempered) distribution of order 0; see [Chapters 6 and 7] of [4]. Notably, its Borel-measure counterpart is the *Dirac measure*

$$\delta : E \mapsto \big[0 \in E \big] \tag{A.62}$$

restricted to Borel sets in $\mathbf{R}$. This yields the special case of (28):

$$\int_{\mathbf{R}} \phi \, d\delta = \delta(\phi) = \phi(0). \tag{A.63}$$

Convolution of δ with the translated signal $\phi_t : s \mapsto \phi(t - s)$ extends (A.64), as follows:

$$[\delta * \phi](t) \triangleq \int_{\mathbf{R}} \phi_t \, d\delta = \delta(\phi_t) = \phi(t). \tag{A.64}$$

The Radon measure δ now serves as the convolution identity.

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